

Algebraic bosonization: the study of the Heisenberg and Calogero-Sutherland models

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Abstract

We propose an approach to treat $(1+1)$ -dimensional fermionic systems based on the idea of *algebraic bosonization*. This amounts to decompose the elementary low-lying excitations around the Fermi surface in terms of basic building blocks which carry a representation of the $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra, which is the dynamical symmetry of the Fermi quantum incompressible fluid. This symmetry simply expresses the local particle-number current conservation at the Fermi surface. The general approach is illustrated in detail in two examples: the Heisenberg and Calogero-Sutherland models, which allow for a comparison with the exact Bethe Ansatz solution.

March 1996

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1 Introduction

There are many $(1 + 1)$ -dimensional models of non-relativistic fermions which are of contemporary interest, either for theoretical reasons, or because of their applicability to specific condensed matter or statistical systems (for modern introductions to this subject see for example Refs. [1, 2]). Some of these models admit an exact solution by the application of the Bethe Ansatz technique [3] (for a general review see, for example, Ref. [4]). Although this method is very powerful and provides a deep conceptual insight that most approximations miss, in some cases it may be difficult to extract explicit results from it. Moreover, there are many interesting fermionic systems that cannot be exactly solved by the Bethe Ansatz.

In this paper, we present a method that can partially circumvent these problems leading to simple and tractable expressions, at least in perturbation theory, and which can be also applied to non-integrable models. We shall call this procedure *algebraic bosonization*, which can be described as a sequence of simple steps. The first is to identify the Fermi surface (the set of left and right Fermi points in the simplest case) of the $(1 + 1)$ -dimensional systems under consideration, and then study its small fluctuations [5, 6, 7], *i.e.* those many-body configurations of low momenta in the vicinity of the Fermi surface. The dynamics of such configurations is governed by an effective hamiltonian \mathcal{H} which is, in general, simpler than the original one. In the thermodynamic limit of a large number of fermions N at constant density ρ , \mathcal{H} scales as a power series in $1/N$. As a consequence, only the first few terms of this expansion must be considered to achieve a given degree of accuracy. In particular, for a gapless system, *i.e.* with a linear dispersion relation around the Fermi points, the $1/N$ term of the effective hamiltonian identifies a conformal field theory [8]. In fact, one has

$$\mathcal{H}_{(1/N)} = \frac{2\pi}{N} \rho v \left(L_0 + \overline{L}_0 \right) \quad (1.1)$$

where v is the Fermi velocity, and L_0 and \overline{L}_0 are the zero modes of the right and left Virasoro (conformal) algebras. The validity of Eq. (1.1) has been established for the general class of the so-called Luttinger systems [5, 2], and for all gapless models solvable by the Bethe Ansatz [9, 10, 11].

The spectrum of $\mathcal{H}_{(1/N)}$ follows directly from the representation theory of the Virasoro algebra, which appears as the dynamical symmetry (or spectrum generating algebra) of the effective hamiltonian to order $1/N$. At this point, it is natural to ask whether a dynamical symmetry exists even if the subleading $O(1/N^2)$ -terms of \mathcal{H} are taken into account. Given that these terms originate partly from the interactions among the fermions, and partly from the non-linearity of the dispersion curve around the Fermi surface, the Virasoro algebra is insufficient to describe this new situation and some extension of it becomes necessary.

To get a hint of what the new algebra might be, one can note the following observations. In the thermodynamic limit, the dynamics of the zero-dimensional Fermi surface becomes semiclassical [5, 6, 7] and the one-dimensional Fermi sea behaves as a droplet of an incompressible classical fluid in momentum space. Obviously, this can be thought also as a one-dimensional section of a two-dimensional incompressible droplet. The classical configurations of the latter are characterized by the dynamical symmetry under the area-preserving diffeomorphisms, which generate the so-called w_∞ algebra [12]. In view of these considerations, we are led to propose as dynamical symmetry of the effective hamiltonian the $W_{1+\infty}$ algebra [12, 13], which is a quantum version of the w_∞ algebra generated by the small fluctuations of the Fermi surface of the $(1+1)$ -dimensional system. For the systems we shall consider, $W_{1+\infty}$ simply expresses the local conservation of the particle-number current at each Fermi point.

This situation resembles the physics of the quantum Hall effect (for a review see, *e.g.* Ref. [14]), where the role of the Fermi sea (in configuration space) is taken by the Laughlin's quantum incompressible fluid [15]. Indeed, the latter has been shown to possess the $W_{1+\infty}$ dynamical symmetry generated by the edge excitations [16], which can be bosonized using the $W_{1+\infty}$ symmetry [17]. The $W_{1+\infty}$ algebra is a linear and infinite dimensional extension of the Virasoro algebra, containing generators V_n^i (with $n \in \mathbf{Z}$ and $i \geq 0$) of arbitrary integer conformal spin $i+1$ (in this notation, the standard Virasoro generators L_n are denoted by V_n^1).

The second step of our procedure is to show that the complete effective hamiltonian \mathcal{H} of the fermionic system displays a $W_{1+\infty}$ structure. This amounts to prove that the subleading part in the $1/N$ expansion can be written entirely in terms of V^i currents. If this condition is met, then the Hilbert space of the effective hamiltonian is described by a set of unitary, irreducible, highest-weight representations of the $W_{1+\infty}$ algebra, which are known and completely classified [13]. Hence the spectrum of the low-lying excitations can be readily obtained.

We would like to stress that the purpose of this procedure is not to simply rewrite the effective hamiltonian in a different fashion, but instead to extend to all orders in $1/N$ the abelian bosonization of the Luttinger models [18, 19, 20, 5, 2]. In fact, if the fundamental degrees of freedom of the effective theory are the modes of the currents V^i , then any realization of these can be chosen for convenience. In particular, a realization of the $W_{1+\infty}$ algebra in terms of a bosonic field can then be used to describe the system in place of the original fermionic degrees of freedom. In this way, one can take advantage of the existence a free parameter, the compactification radius of the bosonic field, to diagonalize the *entire* effective hamiltonian.

In this paper we illustrate in detail this procedure in two different systems: the Heisenberg and the Calogero-Sutherland models. The former can be mapped by means of a Jordan-Wigner transformation into a theory of fermions on a lattice with

a short-range interaction; the latter is, instead, a continuum theory with long-range interactions. Despite this difference at the microscopic level, the effective hamiltonians \mathcal{H} of the two models turn out to have the same structure, namely that of an interacting Luttinger liquid [5, 2]. By exposing the $W_{1+\infty}$ structure of \mathcal{H} , we will be able to use the $W_{1+\infty}$ representation theory to find the spectrum of their low-energy excitations.

The paper is organized as follows. In Section 2 we derive the effective theory of the Heisenberg and Calogero-Sutherland models, using spinless fermions as microscopic degrees of freedom. In Section 3 we give a brief summary of the main properties of the $W_{1+\infty}$ algebra, and present explicitly both its fermionic and bosonic realizations which will be useful in the sequel; some technical and mathematical details on the $W_{1+\infty}$ algebra and its representations are, instead, collected in the Appendix. In Section 4 we study the $W_{1+\infty}$ structure of the Calogero-Sutherland model, and compare our results with those obtained from the exact Bethe Ansatz solution. We also comment on the connections between the two methods. In Section 5 we apply the same procedure to the Heisenberg model, up to order $1/N^2$ in the presence of an external magnetic field B , and up to order $1/N^3$ when $B = 0$. Finally, in Section 6 we present our conclusions.

2 The effective theory of the Heisenberg and the Calogero-Sutherland models

In this section we will derive the effective hamiltonian for the Heisenberg and the Calogero-Sutherland models in $(1+1)$ -dimensions, and show that both of them have the structure of an interacting Luttinger liquid [5, 2]. This result is achieved by using spinless fermions as microscopic degrees of freedom, and linearizing their dispersion relations around the Fermi points. We begin by discussing in detail this procedure for the Heisenberg model, leaving the analysis of the Calogero-Sutherland model for later.

The Heisenberg model

The Heisenberg model describes the exchange interactions among spins localized on the sites of a lattice, with hamiltonian

$$H = \sum_{\langle j,k \rangle} J \mathbf{S}_j \cdot \mathbf{S}_k = \sum_{\langle j,k \rangle} \left[\frac{J}{2} (S_j^+ S_k^- + S_j^- S_k^+) + J S_j^z S_k^z \right] . \quad (2.1)$$

Here J is a coupling constant and the symbol $\langle j, k \rangle$ denotes, as usual, a pair of nearest-neighbor sites. The spin operators $\mathbf{S}_j \equiv \{S_j^+, S_j^-, S_j^z\}$ close the $SU(2)$ algebra

$$\begin{aligned} [S_j^+, S_k^-] &= 2 S_j^z \delta_{j,k} \quad , \\ [S_j^\pm, S_k^z] &= \pm S_j^\pm \delta_{j,k} \quad , \end{aligned} \quad (2.2)$$

and belong to the spin s representation, *i.e.* $\mathbf{S}_j \cdot \mathbf{S}_j = s(s+1)$ for all j . In the following we will consider the case $s = 1/2$ only. Of particular interest is the deformation of the model (2.1) known as the XXZ model [24]. This is characterized by a spin anisotropy in the z direction encoded in the difference between the values of the coupling constants of the two terms of the square brackets of Eq. (2.1).

For a one dimensional chain with N sites, the hamiltonian of the XXZ model in an external magnetic field B is

$$H = H_{XX} + H_Z + H_B \quad (2.3)$$

with

$$\begin{aligned} H_{XX} &= \frac{J}{2} \sum_{j=1}^N (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \quad , \\ H_Z &= J_z \sum_{j=1}^N S_j^z S_{j+1}^z \quad , \\ H_B &= -B \sum_{j=1}^N S_j^z \quad , \end{aligned} \quad (2.4)$$

where $J_z (\neq J)$ is the coupling constant that gives the anisotropy in the z direction. We will assume periodic boundary conditions on the chain, *i.e.* $\mathbf{S}_{N+1} \equiv \mathbf{S}_1$, and set $J = -1$ in H_{XX} for convenience. The model (2.3) can be exactly solved by the Bethe Ansatz for general values of the coupling constant J_z [24], and a large amount of physical information can be obtained from this solution [4]. In particular one finds that the system is gapless in the antiferromagnetic regime ($J_z > -1$) with low-lying excitations above the ground state described by a two-dimensional conformal field theory. This theory captures the universal physical behavior of the system at large distances, such as the critical exponents, and hence it must be regarded as the effective theory corresponding to the original microscopic model. In what follows, we will study this conformal field theory and its $W_{1+\infty}$ algebra extension. To do so, we will follow essentially the same strategy outlined in the seminal paper by Luther and Peschel [20]: we will first transform the Heisenberg model (2.4) into a system of interacting spinless fermions with a linear dispersion relation near the Fermi surface, and then bosonize it.

The first step is easily achieved: we can obtain a fermionic description of the spin-1/2 Heisenberg model by means of the Jordan-Wigner transformation. In fact, if we

introduce spinless fermionic oscillators, $\hat{\psi}_j$, with standard anticommutation relations

$$\begin{aligned}\{\hat{\psi}_j, \hat{\psi}_k\} &= \{\hat{\psi}_j^\dagger, \hat{\psi}_k^\dagger\} = 0 \quad , \\ \{\hat{\psi}_j, \hat{\psi}_k^\dagger\} &= \delta_{j,k} \quad ,\end{aligned}\tag{2.5}$$

then, the algebra (2.2) can be identically satisfied by defining

$$\begin{aligned}S_j^+ &= \exp\left(i\pi \sum_{k=1}^{j-1} n_k\right) \hat{\psi}_j^\dagger \quad , \\ S_j^- &= \exp\left(-i\pi \sum_{k=1}^{j-1} n_k\right) \hat{\psi}_j \quad , \\ S_j^z &= \hat{\psi}_j^\dagger \hat{\psi}_j - \frac{1}{2} \quad ,\end{aligned}\tag{2.6}$$

where $n_j \equiv \hat{\psi}_j^\dagger \hat{\psi}_j$ is the fermion number at site j . The operators $\exp\left(\pm i\pi \sum_{k=1}^{j-1} n_k\right)$ are cocycle factors that must be introduced to correct the fermionic statistics of $\hat{\psi}_j$ and $\hat{\psi}_j^\dagger$, in such a way that S_j^\pm at different sites commute with each other, as required by Eq. (2.2).

To properly write the hamiltonian H in the fermionic representation, we observe that the expectation value of S_j^z in the antiferromagnetic ground state, $\langle S_j^z \rangle$, is not vanishing in the presence of a magnetic field. Thus, in general we can write

$$S_j^z = : \hat{\psi}_j^\dagger \hat{\psi}_j : + \frac{\sigma}{2} \quad ,\tag{2.7}$$

where colons denote the normal ordering with respect to the ground state, and σ the magnetization per site.

Using this definition and Eq. (2.6), we see that the three terms of Eq. (2.4) become

$$H_{XX} = -\frac{1}{2} \sum_{j=1}^N \left(: \hat{\psi}_j^\dagger \hat{\psi}_{j+1} : + : \hat{\psi}_{j+1}^\dagger \hat{\psi}_j : \right) \quad ,\tag{2.8}$$

$$H_Z = J_z \sum_{j=1}^N \left(: \hat{\psi}_j^\dagger \hat{\psi}_j : - : \hat{\psi}_{j+1}^\dagger \hat{\psi}_{j+1} : \right) + J_z \sigma \sum_{j=1}^N : \hat{\psi}_j^\dagger \hat{\psi}_j : \quad ,\tag{2.9}$$

$$H_B = -B \sum_{j=1}^N : \hat{\psi}_j^\dagger \hat{\psi}_j : \quad ,\tag{2.10}$$

where we have also normal-ordered H_{XX} and dropped irrelevant (finite) constants¹. The total hamiltonian $H = H_{XX} + H_Z + H_B$ describes a system of interacting spinless

¹Notice that the introduction of the normal ordering in H_{XX} yields a finite constant which is the ground state energy of the XX model.

fermions, and consists of terms that are quadratic and quartic in the $\hat{\psi}$'s. Thus, it is useful to distinguish between them and write

$$H = H_0 + H_I \quad , \quad (2.11)$$

where H_0 denotes the quadratic part and H_I the four-fermion interaction. Our attitude will be to focus first on H_0 and then treat H_I as a perturbation. Notice, however, that the “free” hamiltonian H_0 actually depends both on the coupling constant J_z and on the magnetic field B , since both H_Z and H_B contribute to H_0 .

From now on, we will assume for simplicity that the chain has an even number of sites and a unit lattice spacing. Thus, if L denotes the total length of the chain, we have

$$L = N = 2M \quad ,$$

with M being a positive integer number. The Fourier transform of the fermion $\hat{\psi}_j$ is then given by

$$\hat{\psi}_j = \frac{1}{\sqrt{N}} \sum_n \psi_n e^{i k_n j} \quad , \quad (2.12)$$

where $k_n \equiv (2\pi n/N)$ is the momentum of the mode ψ_n . To ensure the appropriate boundary conditions, the index n must be integer if M is odd, and half-integer if M is even. Furthermore, due to the periodicity of the lattice, the sum over n in Eq. (2.12) must be restricted to the first Brillouin zone. In the following, without any loss of generality, we will consider only the case M odd.

Inserting Eq. (2.12) into Eqs. (2.8)-(2.10) and decomposing the result according to Eq. (2.11), we get

$$H_0 = - \sum_{n=-M+1}^M [f(n) + (B - J_z \sigma)] : \psi_n^\dagger \psi_n : \quad , \quad (2.13)$$

$$\begin{aligned} H_I = & \frac{J_z}{N} \sum_{n,n',m,m'=-M+1}^M f(n-n') : \psi_{n'}^\dagger \psi_n : : \psi_{m'}^\dagger \psi_m : \delta(n' - n + m' - m) \\ & + \frac{J_z}{N} \sum_{n,n',m,m'=-M+1}^M e^{i \frac{2\pi}{N}(n'-n)} : \psi_{n'}^\dagger \psi_n : : \psi_{m'}^\dagger \psi_m : \\ & \times \left[\delta(n' - n + m' - m - N) + \delta(n' - n + m' - m + N) \right] \quad , \end{aligned} \quad (2.14)$$

where

$$f(n) = \cos \left(\frac{2\pi}{N} n \right) \quad , \quad (2.15)$$

and $\delta(n)$ stands for the Kronecker delta $\delta_{n,0}$. The last two lines of Eq. (2.14) are the Umklapp terms, which are characterized by the fact that the four momenta of the fermions add up to $\pm 2\pi$, and thus satisfy momentum conservation modulo a reciprocal lattice vector.

When $B = 0$ (and hence $\sigma = 0$), H_0 simply describes a free fermionic system with ground state given by

$$|\Omega\rangle_0 = \psi_{-n_F^0}^\dagger \cdots \psi_{n_F^0}^\dagger |0\rangle \quad ,$$

where $|0\rangle$ is the Fock vacuum of the fermionic oscillators, and $n_F^0 = (M - 1)/2$ is the Fermi point. The introduction of a (not too strong) magnetic field B does not destroy this structure, merely shifting the Fermi level, $n_F^0 \rightarrow n_F$. More precisely, using Eq. (2.7) one finds that

$$n_F = \frac{M}{2}(1 + \sigma) - \frac{1}{2} \quad , \quad (2.16)$$

so that the ground state of H_0 in the presence of a magnetic field is

$$|\Omega\rangle = \psi_{-n_F}^\dagger \cdots \psi_{n_F}^\dagger |0\rangle \quad . \quad (2.17)$$

The quantity $n_F/(N/2)$ is usually called the filling factor [1], which in the absence of an external magnetic field takes the value $1/2$ in the thermodynamic limit. The two isolated points $\pm n_F$ form the Fermi surface of the system. For convenience here and in the following, we assume that the quantity $(M\sigma/2)$ is an integer in such a way that n_F is simply obtained from n_F^0 with an integer shift ². Later on, we will determine the precise relation between σ and the magnetic field B (see Eq. (5.1)), but for the time being this relation is not necessary. We only notice here that the shift of the Fermi surface induced by the magnetic field guarantees that the Umklapp terms do not contribute to the low-energy effective hamiltonian, as we will see momentarily.

In general, only the oscillators near the Fermi points $\pm n_F$ play an important role in physical processes. In fact, they produce the low-energy excitations above the ground state $|\Omega\rangle$ and determine the large-distance properties of the system which are described by the effective theory. In order to write the hamiltonian for this effective theory, we define shifted fermionic operators associated to the small fluctuations around each Fermi point according to

$$\begin{aligned} a_r &\equiv \psi_{n_F+r} & , & & a_r^\dagger &\equiv \psi_{n_F+r}^\dagger & , \\ b_r &\equiv \psi_{-n_F-r} & , & & b_r^\dagger &\equiv \psi_{-n_F-r}^\dagger & . \end{aligned} \quad (2.18)$$

The quantity $2\pi(r - 1/2)/N$ ($-2\pi(r - 1/2)/N$) represents the momentum of the oscillator a_r (b_r) relative to the right (left) Fermi point. The integer index r can only vary in a finite range, say between $-\Lambda_0$ and $+\Lambda_0$ where Λ_0 is a bandwidth cut-off. We choose it such that $\Lambda_0 \ll n_F$, and $\Lambda_0 = o(n_F) = o(N)$ in the thermodynamic limit $N \rightarrow \infty$. Roughly speaking, Λ_0 indicates how far from the Fermi points one can go without leaving the effective regime. Clearly the oscillators a and b in Eq. (2.18) form

²Notice that this requirement implies that σ has to be quantized in units of $2/M = 1/N$, but the effects of this discretization actually disappear in the thermodynamic limit $N \rightarrow \infty$.

two independent sets and define two independent and finite branches of excitations, one around the left and one around the right Fermi points. They are such that

$$\begin{aligned} a_r|\Omega\rangle &= 0 & , & & b_r|\Omega\rangle &= 0 & \text{ for } r = 1, 2, \dots, \Lambda_0 & , \\ a_s^\dagger|\Omega\rangle &= 0 & , & & b_s^\dagger|\Omega\rangle &= 0 & \text{ for } s = 0, -1, -2, \dots, -\Lambda_0 & . \end{aligned} \quad (2.19)$$

The procedure to find the effective hamiltonian \mathcal{H} corresponding to H is now simple. The first step is to select all terms in H containing oscillators whose index lies in the range $[-\Lambda_0, +\Lambda_0]$ around each Fermi point. For example, for the quadratic part of the hamiltonian, H_0 , we get

$$\mathcal{H}_0 = - \sum_{r=-\Lambda_0}^{\Lambda_0} \left[f(r + n_F) + (B - J_z \sigma) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) , \quad (2.20)$$

where the normal ordering is defined with respect to $|\Omega\rangle$ according to Eq. (2.19). After using Eqs. (2.15) and (2.16), we expand the right hand side of Eq. (2.20) in powers of $1/N$ to obtain

$$\begin{aligned} \mathcal{H}_0 &= \left(-B + J_z \sigma + \sin \frac{\pi \sigma}{2} \right) \sum_{r=-\Lambda_0}^{\Lambda_0} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &+ \frac{2\pi}{N} \cos \frac{\pi \sigma}{2} \sum_{r=-\Lambda_0}^{\Lambda_0} \left(r - \frac{1}{2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &- \frac{1}{2} \left(\frac{2\pi}{N} \right)^2 \sin \frac{\pi \sigma}{2} \sum_{r=-\Lambda_0}^{\Lambda_0} \left(r - \frac{1}{2} \right)^2 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &- \frac{1}{6} \left(\frac{2\pi}{N} \right)^3 \cos \frac{\pi \sigma}{2} \sum_{r=-\Lambda_0}^{\Lambda_0} \left(r - \frac{1}{2} \right)^3 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &+ O\left(\frac{1}{N^4} \right) . \end{aligned} \quad (2.21)$$

We remark that this expansion is meaningful because the sum over r has a finite range and N is assumed to be large. The thermodynamic limit is correctly defined because, according to our assumptions, $\Lambda_0 = o(N)$ when $N \rightarrow \infty$. In particular, we stress that the $1/N$ -term of Eq. (2.21) has a linear dispersion relation, signaling the fact that the system is gapless. The $O(1/N^2)$ -terms of this expansion are higher order corrections and will be analyzed in the following sections using the structure of the $W_{1+\infty}$ algebra.

Let us now turn to the interaction term, Eq. (2.14). To write the corresponding effective hamiltonian \mathcal{H}_I , we distinguish among three cases:

1. when the momentum exchanged in the interaction is small, that is $|n' - n| \sim 0$;

2. when the exchanged momentum is roughly twice the Fermi momentum, that is $|n' - n| \sim 2n_F$;
3. when Umklapp processes take place.

The first case corresponds to a forward scattering, whilst the second to a backward scattering; both kinds of processes are described by the first line of Eq. (2.14). The Umklapp terms instead, originate only from the last two lines of Eq. (2.14). According to this classification, it is natural to decompose the effective hamiltonian as follows

$$\mathcal{H}_I = \mathcal{H}_{forw} + \mathcal{H}_{back} + \mathcal{H}_{Umk} \quad .$$

Let us first consider the forward scattering terms. These arise from Eq. (2.14) with the following choice of configuration of fermionic indices

$$\begin{aligned} n &= \pm(n_F + r) & , & & n' &= n - \ell & , \\ m &= \pm(n_F + s) & , & & m' &= m + \ell & , \end{aligned} \quad (2.22)$$

with

$$\begin{aligned} |r| &\leq \Lambda_0 & , & & |s| &\leq \Lambda_0 & , \\ |r \mp \ell| &\leq \Lambda_0 & , & & |s \pm \ell| &\leq \Lambda_0 & . \end{aligned} \quad (2.23)$$

In Eq. (2.22) we can take either sign independently, so that we find

$$\begin{aligned} \mathcal{H}_{forw} &= \frac{J_z}{N} \sum_{\ell}' \sum_{r,s=-\Lambda_0}^{\Lambda_0} f(\ell) \left[:a_{r-\ell}^{\dagger} a_r :: a_{s+\ell}^{\dagger} a_s : + :b_{r+\ell}^{\dagger} b_r :: b_{s-\ell}^{\dagger} b_s : \right. \\ &\quad \left. + :a_{r-\ell}^{\dagger} a_r :: b_{s-\ell}^{\dagger} b_s : + :b_{r+\ell}^{\dagger} b_r :: a_{s+\ell}^{\dagger} a_s : \right] . \end{aligned} \quad (2.24)$$

Here the symbol \sum' means that the sum over ℓ is restricted to those values that satisfy the constraints (2.23). Of the four terms appearing in the square bracket of Eq. (2.24), the first two represent the forward scattering among four particles belonging to the same branch of the dispersion curve (right or left), while the second two represent the forward scattering between two pairs of particles belonging to different branches.

Next, we examine the backward scattering part of the effective hamiltonian. It corresponds to the following configurations of indices for the fermions in Eq. (2.14)

$$\begin{aligned} n &= \pm(n_F + r) & , & & n' &= n \mp 2n_F - \ell & , \\ m &= \mp(n_F + s) & , & & m' &= m \pm 2n_F + \ell & , \end{aligned} \quad (2.25)$$

where, once again, r , s and ℓ satisfy Eq. (2.23). To fulfill momentum conservation, we must take in Eq. (2.25) either all upper or all lower signs, thus generating two

different structures in the effective hamiltonian. In fact, we find

$$\begin{aligned} \mathcal{H}_{back} = & \frac{J_z}{N} \sum_{\ell}' \sum_{r,s=-\Lambda_0}^{\Lambda_0} \left[f(2n_F + \ell) b_{-r+\ell}^{\dagger} a_r a_{-s+\ell}^{\dagger} b_s \right. \\ & \left. + f(-2n_F + \ell) a_{-r-\ell}^{\dagger} b_r b_{-s-\ell}^{\dagger} a_s \right] . \end{aligned} \quad (2.26)$$

Finally, we consider the Umklapp terms, which, as mentioned above, originate from the last two lines of Eq. (2.14). These terms are important only if the band is half-filled, in which case all four fermions can be near the Fermi surface. If the band is not half-filled, the Umklapp processes do not contribute to the effective hamiltonian. This is precisely what happens for the Heisenberg model in the presence of a magnetic field B . In fact, according to Eq. (2.16) for $\sigma \neq 0$, the difference $(4n_F - N)$ is of order N in the thermodynamic limit. Therefore, the delta functions in the last line of Eq. (2.14) cannot have vanishing argument if n' , n , m' and m differ from $\pm n_F$ by a quantity $|\ell| \leq 2\Lambda_0 = o(N)$ for $N \rightarrow \infty$. This statement is true for any non-zero magnetic field B , even if very small. Thus, in the following, we will always neglect the Umklapp contributions, taking

$$\mathcal{H}_{Umk} = 0 \quad .$$

Notice that the case with no magnetic field also shares this property provided it is defined as the limit $B \rightarrow 0$ of the case with non-vanishing B .

The next step to find the low-energy theory is the crucial one: we remove the bandwidth cut-off Λ_0 by sending it to infinity, as in the mapping of the Tomonaga model [25] into the Luttinger model [18]. However, in order to avoid the introduction of spurious low-energy states (the $O(1/N^2)$ -corrections do bend the dispersion curve), we will keep always $\Lambda_0 \ll N$. For ease of notation, when Λ_0 and $N \rightarrow \infty$, we will write the free effective hamiltonian \mathcal{H}_0 simply as in Eq. (2.21) with all sums extended from $-\infty$ to $+\infty$ but with the limit $N \rightarrow \infty$ left implicit. Even though \mathcal{H}_0 now contains new oscillators, it should be clear that it still acts on low-energy states only, with particle and hole momenta bounded by $\Lambda_0 \ll N$. Nonetheless, the extension of the dispersion curve to infinity is not free of consequences. In fact, since now Eqs. (2.19) hold for *any* integer r and s , the ground state $|\Omega\rangle$ corresponds to the surface of two *infinite* Fermi seas (one left and one right). This fact will play a crucial role when describing the algebraic properties of the effective theory, leading to an easier and more elegant mathematical interpretation, as we shall see momentarily.

When $\Lambda_0 \rightarrow \infty$, it is natural to define the following two chiral Weyl fermions

$$F_+(\theta) = \sqrt{\frac{2\pi}{N}} \sum_{r=-\infty}^{\infty} a_r e^{i(r-\frac{1}{2})\theta} \quad , \quad (2.27)$$

and

$$F_-(\theta) = \sqrt{\frac{2\pi}{N}} \sum_{r=-\infty}^{\infty} b_r e^{i(r-\frac{1}{2})\theta} \quad . \quad (2.28)$$

Since the index r is integer, these fields satisfy antiperiodic boundary conditions on a circle of radius $R = N/2\pi$, parametrized by the angle $\theta \in [0, 2\pi)$. Using F_+ and F_- , it is straightforward to check that the $1/N$ -term in \mathcal{H}_0 can be written as

$$\mathcal{H}_0 \Big|_{1/N} = \frac{1}{2\pi} \int_0^{2\pi} d\theta : \left[F_+^\dagger (-i\partial_\theta) F_+ + F_-^\dagger (-i\partial_\theta) F_- \right] : \quad . \quad (2.29)$$

This is the Dirac hamiltonian for a free relativistic fermion in $(1+1)$ -dimensions given by

$$\Phi = \begin{pmatrix} F_+ \\ F_- \end{pmatrix} \quad .$$

The other terms of \mathcal{H}_0 can also be nicely written in terms of this Dirac field. As it is well-known, in $(1+1)$ -dimensions a Dirac fermion is equivalent to a scalar boson through the abelian bosonization procedure. Thus, the free hamiltonian \mathcal{H}_0 can be given an equivalent description using bosons instead of fermions. Actually, we shall see that this is possible even for the interaction hamiltonian \mathcal{H}_I .

To do so, we must analyze \mathcal{H}_{forw} and \mathcal{H}_{back} when the bandwidth is extended to infinity. For the forward scattering part, Eq. (2.24) simply leads to

$$\begin{aligned} \mathcal{H}_{forw} &= \frac{J_z}{N} \sum_{\ell, r, s=-\infty}^{\infty} f(\ell) \left[: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : \right. \\ &\quad \left. + 2 : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right] \quad , \end{aligned} \quad (2.30)$$

where we have used the property $f(\ell) = f(-\ell)$. Since all terms in the r.h.s. of Eq. (2.30) are already normal ordered in each pair of fermions, this equation is a good starting point to implement the bosonization procedure, as we shall discuss in the following sections.

The backward scattering terms, Eq. (2.26), require more attention instead. In fact, before one can bosonize them, it is necessary to rearrange the a and b oscillators to reconstruct normal ordered pairs. Sending $\Lambda_0 \rightarrow \infty$, using the property $f(\ell \pm N/2) = -f(\ell)$ and then relabeling the summation index ℓ , we can rewrite \mathcal{H}_{back} as follows

$$\mathcal{H}_{back} = -\frac{J_z}{N} \sum_{\ell, r, s=-\infty}^{\infty} f(\ell) \left[b_{-r+\ell+\gamma}^\dagger a_r a_{-s+\ell+\gamma}^\dagger b_s + a_{-r-\ell+\gamma}^\dagger b_r b_{-s-\ell+\gamma}^\dagger a_s \right] \quad , \quad (2.31)$$

where the integer γ is defined by

$$\gamma = 1 - \frac{N\sigma}{2} = \frac{N}{2} - 2n_F \quad . \quad (2.32)$$

In the square brackets of Eq. (2.31) we have collected together two terms that describe backward scattering processes involving the same exchanged momentum; in fact, according to Eq. (2.32),

$$k = \frac{2\pi}{N} (2n_F + \ell + \gamma) \quad ,$$

which is the momentum exchanged by the first term, and

$$k' = \frac{2\pi}{N} (-2n_F + \ell - \gamma) \quad ,$$

which is the momentum exchanged by the second term, differ by 2π . Guided by this observation, we normal order \mathcal{H}_{back} without breaking the square bracket of Eq. (2.31); this assures that no divergences appear. Indeed, after performing standard manipulations and dropping irrelevant additive constants, we find

$$\mathcal{H}_{back} = \mathcal{H}_{back}^{(4)} + \mathcal{H}_{back}^{(2)}$$

where the four- and two-fermion parts are given, respectively, by

$$\mathcal{H}_{back}^{(4)} = 2 \frac{J_z}{N} \sum_{\ell, r, s=-\infty}^{\infty} f(\ell) : a_{-s+\ell+\gamma}^\dagger a_r : : b_{-r+\ell+\gamma}^\dagger b_s : \quad , \quad (2.33)$$

and

$$\mathcal{H}_{back}^{(2)} = \frac{J_z}{N} \sum_{\ell=-\infty}^{\infty} f(\ell) \left[\left(\sum_{r=|\ell|+\gamma}^{\infty} - \sum_{r=-\infty}^{-|\ell|+\gamma-1} \right) (: a_r^\dagger a_r : + : b_r^\dagger b_r :) \right] \quad . \quad (2.34)$$

Since the oscillators are already normal ordered, it is safe to shift their indices, and also to exchange the order of the sums. For example, if in Eq. (2.33) we let $\ell \rightarrow r + s - \ell - \gamma$, we get

$$\begin{aligned} \mathcal{H}_{back}^{(4)} &= 2 \frac{J_z}{N} \sum_{\ell, r, s=-\infty}^{\infty} f(r + s - \ell - \gamma) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \\ &= 2 \frac{J_z}{N} \sum_{\ell, r, s=-\infty}^{\infty} \left[\cos \pi \sigma \cos \frac{2\pi}{N} (r + s - \ell - 1) \right. \\ &\quad \left. - \sin \pi \sigma \sin \frac{2\pi}{N} (r + s - \ell - 1) \right] : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad , \end{aligned} \quad (2.35)$$

where in the last step we have used Eqs. (2.15) and (2.32). These manipulations have rendered the operator structure of $\mathcal{H}_{back}^{(4)}$ identical to that of the last term of \mathcal{H}_{forw} in Eq. (2.30), the only difference remaining in the numerical function in front.

It is convenient to simplify also the two-fermion part of the backscattering hamiltonian. To do so, we first exchange the sums over ℓ and r in Eq. (2.34), and then exploit the following formula

$$\sum_{\ell=1}^n f(\ell) = \sum_{\ell=1}^n \cos \left(\frac{2\pi}{N} \ell \right) = \frac{1}{2} \left[\frac{\sin \frac{2\pi}{N} \left(n + \frac{1}{2} \right)}{\sin \frac{\pi}{N}} - 1 \right] \quad .$$

After some straightforward algebra, we find

$$\begin{aligned}
\mathcal{H}_{back}^{(2)} &= \frac{J_z}{N} \frac{1}{\sin \frac{\pi}{N}} \sum_{r=-\infty}^{\infty} \sin \frac{2\pi}{N} \left(r - \gamma + \frac{1}{2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&= \frac{J_z}{N} \frac{1}{\sin \frac{\pi}{N}} \sum_{r=-\infty}^{\infty} \left[\cos \pi \sigma \sin \frac{2\pi}{N} \left(r - \frac{1}{2} \right) \right. \\
&\quad \left. + \sin \pi \sigma \cos \frac{2\pi}{N} \left(r - \frac{1}{2} \right) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad .
\end{aligned} \tag{2.36}$$

We now summarize our results by collecting all terms of the effective hamiltonian \mathcal{H} ; for future reference, we organize them as power series in $1/N$ according to

$$\mathcal{H} = \sum_{k=0}^{\infty} \left(\frac{2\pi}{N} \right)^k \mathcal{H}_{(k)} \quad . \tag{2.37}$$

Using Eqs. (2.21), (2.30), (2.35) and (2.36), the first few terms of this series are

$$\mathcal{H}_{(0)} = \left(-B + J_z \sigma + \sin \frac{\pi \sigma}{2} + \frac{J_z}{\pi} \sin \pi \sigma \right) \sum_{r=-\infty}^{\infty} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad , \tag{2.38}$$

$$\begin{aligned}
\mathcal{H}_{(1)} &= \left(\cos \frac{\pi \sigma}{2} + \frac{J_z}{\pi} \cos \pi \sigma \right) \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&\quad + \frac{J_z}{2\pi} \sum_{\ell, r, s=-\infty}^{\infty} \left(: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : \right) \\
&\quad + \frac{J_z}{\pi} (\cos \pi \sigma + 1) \sum_{\ell, r, s=-\infty}^{\infty} : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad ,
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
\mathcal{H}_{(2)} &= -\frac{1}{2} \left(\sin \frac{\pi \sigma}{2} + \frac{J_z}{\pi} \sin \pi \sigma \right) \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2} \right)^2 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&\quad + \frac{J_z}{24\pi} \sin \pi \sigma \sum_{r=-\infty}^{\infty} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&\quad - \frac{J_z}{\pi} \sin \pi \sigma \sum_{\ell, r, s=-\infty}^{\infty} (r + s - \ell - 1) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad ,
\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}
\mathcal{H}_{(3)} &= -\frac{1}{6} \left(\cos \frac{\pi \sigma}{2} + \frac{J_z}{\pi} \cos \pi \sigma \right) \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2} \right)^3 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&\quad + \frac{J_z}{24\pi} \cos \pi \sigma \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\
&\quad - \frac{J_z}{4\pi} \sum_{\ell, r, s=-\infty}^{\infty} \ell^2 \left(: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : \right) \\
&\quad - \frac{J_z}{2\pi} \sum_{\ell, r, s=-\infty}^{\infty} \left[(r + s - \ell - 1)^2 \cos \pi \sigma + \ell^2 \right] : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad .
\end{aligned} \tag{2.41}$$

Notice that $\mathcal{H}_{(0)}$ and $\mathcal{H}_{(2)}$ vanish when $B = 0$ (and hence $\sigma = 0$). This property actually holds for all terms $\mathcal{H}_{(k)}$ with k even. It is also interesting to observe that the two terms proportional to J_z in $\mathcal{H}_{(0)}$ have a different origin. The first comes directly from the “free” hamiltonian \mathcal{H}_0 of Eq. (2.13), while the second comes from the backward scattering part of the interaction hamiltonian and thus is a normal ordering effect ³.

We conclude our discussion of the Heisenberg model with a few more comments. The effective hamiltonian given in Eqs. (2.37)-(2.41) is similar but not identical to that of Ref. [20] (where only the $1/N$ -term with a vanishing magnetic field is explicitly considered). The difference between our results and those of Ref. [20] is due to a different normal ordering prescription for the interaction hamiltonian \mathcal{H}_I . In our derivation, we have always consistently used the normal ordering as dictated by the Jordan-Wigner transformation. Consequently, the forward scattering part given in Eq. (2.30), is automatically normal ordered in each pair of fermions. The backward scattering part, instead, requires a rearrangement that produces a normal ordered four-fermion piece given in Eq. (2.35), and also a two-body part given in Eq. (2.36). It is precisely the latter in addition to the forward scattering part that causes the difference between our effective hamiltonian and that of Ref. [20]. We will comment more on this fact in Section 5 when we compare our results and the exact Bethe Ansatz solution.

The Calogero-Sutherland model

The Calogero-Sutherland model describes the interaction of N non-relativistic fermions of mass m moving on a circle of length L with a pairwise potential proportional to the inverse square of the chord distance between the two particles [22, 23]. If we denote by x_i the coordinate of the i -th fermion along the circle and choose units such that $2m = 1$, then the hamiltonian is

$$h = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + g \frac{\pi^2}{L^2} \sum_{j < k} \frac{1}{\sin^2(\pi(x_j - x_k)/L)} \quad , \quad (2.42)$$

where g is the coupling constant. In the following, without any loss of generality, we will take N to be odd.

This model can be exactly solved by Bethe Ansatz and all its fundamental properties can be obtained from this solution [23]. In particular, one finds that the low-energy excitations above the ground state are gapless, and thus the long distance properties of the system are described by a conformal field theory [11]. In what fol-

³ As a useful check, let us observe that the effective Hamiltonian \mathcal{H} is invariant under the transformations $B \rightarrow -B$, $\sigma \rightarrow -\sigma$, $a_r \rightarrow a_{1-r}^\dagger$, $b_r \rightarrow b_{1-r}^\dagger$, as the original hamiltonian (2.3) is invariant under $B \rightarrow -B$ and $\mathbf{S}_i \rightarrow -\mathbf{S}_i$.

lows, we will derive explicitly the effective theory of the Calogero-Sutherland model and show that it is identical in structure to that of the Heisenberg model.

The single particle wave functions of h are plane waves

$$\phi_n(x) = \frac{1}{\sqrt{L}} \exp\left(i\frac{2\pi}{L}nx\right) \quad ,$$

with $n \in \mathbf{Z}$ to satisfy periodic boundary conditions. Introducing a set of fermionic oscillators with standard anticommutation relations (cf. Eq. (2.5)), we can define a second quantized non-relativistic fermion field as

$$\Psi(x) = \sum_{n=-\infty}^{\infty} \psi_n \phi_n(x) \quad ,$$

and then compute the second quantized hamiltonian corresponding to h , which is given by

$$H = H_0 + H_I \quad ,$$

where the kinetic part is

$$\begin{aligned} H_0 &= \int_0^L dx \Psi^\dagger(x) \left(-\frac{\partial^2}{\partial x^2}\right) \Psi(x) \\ &= \left(\frac{2\pi}{L}\right)^2 \sum_{n=-\infty}^{\infty} n^2 \psi_n^\dagger \psi_n \quad , \end{aligned} \tag{2.43}$$

while the interaction part is [26]

$$\begin{aligned} H_I &= \frac{1}{2} \int_0^L dx \int_0^L dy \Psi^\dagger(x) \Psi^\dagger(y) \left[g \frac{\pi^2}{L^2} \frac{1}{\sin^2(\pi(x-y)/L)} \right] \Psi(x) \Psi(y) \\ &= -g \frac{\pi^2}{L^2} \sum_{l,n,m=-\infty}^{\infty} |l| \psi_{m+l}^\dagger \psi_{n-l}^\dagger \psi_n \psi_m \quad . \end{aligned} \tag{2.44}$$

The hamiltonian H_0 simply describes a free fermionic system whose ground state $|\Omega\rangle$ is given by Eq. (2.17) with

$$n_F = \frac{N-1}{2} \quad . \tag{2.45}$$

To study the small fluctuations around the Fermi points $\pm n_F$, we define two independent sets of oscillators, a_r and b_r , according to Eq. (2.18). In terms of these degrees of freedom, the kinetic part of the hamiltonian reads

$$\mathcal{H}_0 = \left(\frac{2\pi}{L}\right)^2 \sum_{r=-\Lambda_0}^{\Lambda_0} (n_F + r)^2 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad , \tag{2.46}$$

where the bandwidth cut-off is such that $\Lambda_0 \ll N$ and $\Lambda_0 = o(N)$ in the thermodynamic limit $L, N \rightarrow \infty$, as in the Heisenberg model.

Let us now consider the interaction hamiltonian H_I . Our purpose is to write the corresponding effective operator \mathcal{H}_I in terms of the usual bilinear fermionic forms; once this is done, it is simple to interpret the results within the algebraic context of the extended conformal theories. To do so, however, we need to reorder the oscillators of H_I , for example by moving ψ_m close to ψ_{m+l}^\dagger in Eq. (2.44). In this way, we obtain a four-fermion term that is completely similar in structure to the interaction hamiltonian of the Heisenberg model (cf. Eq. (2.14)), and thus can be treated accordingly. The price we pay to achieve this is the appearance of a divergent two-fermion term. Thus, before we can proceed, it is necessary to introduce a regularization prescription to avoid this divergence and give a meaning to our formulas. Inspired by the procedure followed in our discussion of the Heisenberg model, we perform a “periodic regularization”, namely we divide the momentum space into fictitious Brillouin zones with amplitude $2M$, *i.e.* we identify l and l' if $l' = l + 2Mk$ for any integer k . The number M , serving as a regulator, is taken to be arbitrary with the only constraint $M \gg n_F$ such that the physical region of interest is inside the first zone. Thus, we can perform all calculations involving small oscillations around the Fermi points $\pm n_F$ in the first Brillouin zone, like in the Heisenberg model, and, at the end, let $M \rightarrow \infty$ to recover the original continuum theory. We will now show that this procedure is consistent and leads to finite and meaningful results.

According to this regularization prescription, the hamiltonian H_I of Eq. (2.44) becomes

$$H_I(M) = H'_I(M) + H''_I(M)$$

where

$$\begin{aligned} H'_I(M) = & -g \frac{\pi^2}{L^2} \sum_{n,n',m,m'=-M+1}^M |n-n'|_M : \psi_{n'}^\dagger \psi_n : : \psi_{m'}^\dagger \psi_m : \\ & \times \delta(n' - n + m' - m) \quad , \end{aligned} \quad (2.47)$$

and

$$H''_I(M) = g \frac{\pi^2}{L^2} \left(\sum_{l=-M+1}^M |l|_M \right) \sum_{n=-M+1}^M : \psi_n^\dagger \psi_n : \quad . \quad (2.48)$$

In these equations, all indices have been restricted to the first Brillouin zone and consequently, the absolute value has been replaced by its periodic extension modulo $2M$, which we have denoted by the symbol $| \cdot |_M$. To be specific, we have

$$|\kappa|_M = |\kappa| \quad , \quad |M + \kappa|_M = |-M + \kappa|_M = M - |\kappa| \quad (2.49)$$

for $|\kappa| \leq M$.

The four-fermion part $H'_I(M)$ is identical in form to the first line of Eq. (2.14), and hence the corresponding effective hamiltonian $\mathcal{H}'_I(M)$ contains only a forward

and a backward scattering part ⁴, namely

$$\mathcal{H}'_I(M) = \mathcal{H}_{forw}(M) + \mathcal{H}_{back}(M)$$

The forward scattering terms arise when the indices in Eq. (2.47) satisfy the conditions (2.22) and (2.23), so that

$$\begin{aligned} \mathcal{H}_{forw}(M) = & -g \frac{\pi^2}{L^2} \sum_{\ell}' \sum_{r,s=-\Lambda_0}^{\Lambda_0} |\ell|_M \left[: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : \right. \\ & \left. + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : + : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : + : b_{r+\ell}^\dagger b_r : : a_{s+\ell}^\dagger a_s : \right] . \end{aligned} \quad (2.50)$$

The backward scattering terms, instead, appear when the indices in Eq. (2.47) satisfy the conditions (2.25) and (2.23), so that

$$\begin{aligned} \mathcal{H}_{back}(M) = & -g \frac{\pi^2}{L^2} \sum_{\ell}' \sum_{r,s=-\Lambda_0}^{\Lambda_0} \left[|2n_F + \ell|_M b_{-r+\ell}^\dagger a_r a_{-s+\ell}^\dagger b_s \right. \\ & \left. + | -2n_F + \ell|_M a_{-r-\ell}^\dagger b_r b_{-s-\ell}^\dagger a_s \right] . \end{aligned} \quad (2.51)$$

Finally, let us consider the two-fermion term H''_I of Eq. (2.48). Using the effective degrees of freedom, it becomes

$$\mathcal{H}''_I(M) = g \frac{\pi^2}{L^2} M^2 \sum_{r=-\Lambda_0}^{\Lambda_0} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) . \quad (2.52)$$

Of course, in the limit $M \rightarrow \infty$, $\mathcal{H}''_I(M)$ is divergent. However, when one considers the full hamiltonian, this divergence disappears.

To see this, and also to establish a correspondence with the extended conformal theories, we must remove the bandwidth cut-off. Thus, following the same strategy (and using the same conventions) of the Heisenberg model, we send $N \rightarrow \infty$ and $\Lambda_0 \rightarrow \infty$ in all previous formulas. In this limit, using Eq. (2.45), we easily see that Eq. (2.46) becomes

$$\mathcal{H}_0 = (2\pi\rho_0)^2 \sum_{r=-\infty}^{\infty} \left[\frac{1}{4} + \frac{1}{N} \left(r - \frac{1}{2} \right) + \frac{1}{N^2} \left(r - \frac{1}{2} \right)^2 \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) , \quad (2.53)$$

where $\rho_0 = N/L$ is the density, which is kept fixed in the thermodynamic limit. As in the Heisenberg case, \mathcal{H}_0 is meaningful only when acting on low-energy states with particle and hole momenta bounded by $\Lambda_0 \ll N$.

Analogously, the forward scattering terms, Eq. (2.50), simply turn into

$$\begin{aligned} \mathcal{H}_{forw}(M) = & -\frac{g}{4} (2\pi\rho_0)^2 \frac{1}{N^2} \sum_{\ell,r,s=-\infty}^{\infty} |\ell| \left[: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : \right. \\ & \left. + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : + 2 : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right] . \end{aligned} \quad (2.54)$$

⁴Of course, the Umklapp terms are neither present in the original continuum model, nor produced by our periodic regularization.

Notice that \mathcal{H}_{forw} is independent of the regularization parameter M , and is similar in structure to the forward scattering hamiltonian of the Heisenberg model (cf. Eq. (2.30)).

The backward scattering terms Eq. (2.51) require, instead, more care since a reordering of the a and b oscillators is necessary. Before doing this rearrangement, we remove the bandwidth cut-off and use the definition (2.49) of the periodic modulus to rewrite \mathcal{H}_{back} as follows

$$\begin{aligned} \mathcal{H}_{back}(M) = & -\frac{g}{4}(2\pi\rho_0)^2 \frac{1}{N^2} \sum_{\ell, r, s=-\infty}^{\infty} (M - |\ell|) \left[b_{-r+\ell+\gamma}^\dagger a_r a_{-s+\ell+\gamma}^\dagger b_s \right. \\ & \left. + a_{-r-\ell+\gamma}^\dagger b_r b_{-s-\ell+\gamma}^\dagger a_s \right] \quad , \end{aligned} \quad (2.55)$$

where

$$\gamma = M - N + 1 = M - 2n_F \quad . \quad (2.56)$$

The two terms in the square bracket of Eq. (2.55) describe two backward scattering processes that exchange the same momentum. In fact, $(2n_F + \ell + \gamma)$, which is the momentum exchanged by the first term, and $(-2n_F + \ell - \gamma)$, which is the momentum exchanged by the second term, differ by $2M$, *i.e.* by a period. At this point we can proceed in close analogy to the steps that brought us from Eq. (2.31) to Eqs. (2.35) and (2.36), namely we normal order $\mathcal{H}_{back}(M)$ without breaking the square bracket. After some straightforward algebra, we find

$$\mathcal{H}_{back}(M) = \mathcal{H}_{back}^{(4)}(M) + \mathcal{H}_{back}^{(2)}(M)$$

where

$$\begin{aligned} \mathcal{H}_{back}^{(4)}(M) = & \frac{g}{2}(2\pi\rho_0)^2 \frac{1}{N^2} \\ & \times \sum_{r, s=-\infty}^{\infty} \left[\sum_{\ell=-\infty}^{r+s-M+N-1} (2M - N - r - s + \ell + 1) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right. \\ & \left. + \sum_{\ell=r+s-M+N}^{\infty} (N + r + s - \ell - 1) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right] \quad , \end{aligned} \quad (2.57)$$

and

$$\begin{aligned} \mathcal{H}_{back}^{(2)}(M) = & \frac{g}{4} (2\pi\rho_0)^2 \frac{1}{N^2} \left\{ -M^2 \sum_{r=-\infty}^{\infty} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \right. \\ & + \sum_{r=-\infty}^{\infty} \left[N^2 + N(2r-1) + r(r-1) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ & \left. + 2 \sum_{r=M-N+1}^{\infty} (r-M+N)(r-M+N-1) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \right\} \quad . \end{aligned} \quad (2.58)$$

Since these terms depend explicitly on the regularization parameter M , we must check that they combine to give a finite result when $M \rightarrow \infty$. To this aim, let us consider first the four-fermion operators given by Eqs. (2.54) and (2.57). We have already remarked that $\mathcal{H}_{forw}(M)$ is actually independent of M ; on the contrary $\mathcal{H}_{back}^{(4)}(M)$ does depend on M but, in the limit $M \rightarrow \infty$, it reduces to

$$\mathcal{H}_{back}^{(4)} = \frac{g}{2} (2\pi\rho_0)^2 \frac{1}{N^2} \sum_{\ell, r, s=-\infty}^{\infty} (N + r + s - \ell - 1) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad . \quad (2.59)$$

Indeed, the first term in the r.h.s. of Eq. (2.57) vanishes when $M \rightarrow \infty$, because, given any two states $|v\rangle$ and $|w\rangle$ of the fermionic Fock space, there exists always a positive number k such that

$$\langle v | : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : | w \rangle = 0$$

for any $\ell < -k$. For the same reason, the last term in the r.h.s. of Eq. (2.58) vanishes when $M \rightarrow \infty$. Moreover, the first term exactly cancels with $\mathcal{H}_I''(M)$ of Eq. (2.52). Therefore, if we combine $\mathcal{H}_{back}^{(2)}(M)$ with $\mathcal{H}_I''(M)$ we get a finite result. Indeed,

$$\begin{aligned} \mathcal{H}^{(2)} &\equiv \lim_{M \rightarrow \infty} \left(\mathcal{H}_I''(M) + \mathcal{H}_{back}^{(2)}(M) \right) \\ &= \frac{g}{4} (2\pi\rho_0)^2 \frac{1}{N^2} \sum_{r=-\infty}^{\infty} \left[N^2 + N(2r-1) + r(r-1) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad . \end{aligned} \quad (2.60)$$

We now summarize our results by writing the complete effective hamiltonian of the Calogero-Sutherland model. This is given by

$$\mathcal{H} = (2\pi\rho_0)^2 \sum_{k=0}^2 \frac{1}{N^k} \mathcal{H}_{(k)} \quad (2.61)$$

where

$$\mathcal{H}_{(0)} = \frac{1}{4} (1+g) \sum_{r=-\infty}^{\infty} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad , \quad (2.62)$$

$$\begin{aligned} \mathcal{H}_{(1)} &= \left(1 + \frac{g}{2} \right) \sum_{r=-\infty}^{\infty} \left(r - \frac{1}{2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &\quad + \frac{g}{2} \sum_{\ell, r, s=-\infty}^{\infty} : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad , \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} \mathcal{H}_{(2)} &= \sum_{r=-\infty}^{\infty} \left[\left(r - \frac{1}{2} \right)^2 + \frac{g}{4} (r^2 - r) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &\quad - \frac{g}{4} \sum_{\ell, r, s=-\infty}^{\infty} |\ell| \left[: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : \right. \\ &\quad \left. + 2 : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right] \\ &\quad + \frac{g}{2} \sum_{\ell, r, s=-\infty}^{\infty} (r + s - \ell - 1) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad . \end{aligned} \quad (2.64)$$

Notice that there are no contributions to \mathcal{H} of order $1/N^3$ or higher.

We conclude this section with a few remarks. As in the Heisenberg case, also here the backward scattering part of the interaction hamiltonian plays a crucial role. In fact, in spite of the overall factor $1/L^2$ in Eq. (2.44), it contributes both to the zeroth- and first-order effective hamiltonians $\mathcal{H}_{(0)}$ and $\mathcal{H}_{(1)}$, due to normal ordering effects. On the contrary, the forward scattering part contributes only to $\mathcal{H}_{(2)}$. In Section 4 we will interpret these results in the context of the $W_{1+\infty}$ algebra.

Let us now comment on the general validity of our approach. Both the Heisenberg and Calogero-Sutherland models lead to effective hamiltonians with the same operator structure, which only differ in the functional form of the dispersion and scattering terms. In fact, our method would be equally valid for any other fermionic model of the same form, namely for a hamiltonian with a gapless bilinear kinetic term and a four-fermion interaction term. This class of models is known as the class of generalized Luttinger systems [5, 2, 11, 26]. Note that in our approach there are no special restrictions on the specific form of the dispersion and scattering functions. In particular, no reference to integrability is ever made.

3 The $W_{1+\infty}$ algebraic approach

In the previous section we have seen that the low-energy effective hamiltonian both for the Heisenberg and the Calogero-Sutherland models can be written entirely in terms of fermionic bilinear operators of the generic form

$$\mathcal{O}_\ell = \sum_{r=-\infty}^{\infty} f_\ell(r) : a_{r-\ell}^\dagger a_r : \quad ,$$

or

$$\overline{\mathcal{O}}_\ell = \sum_{r=-\infty}^{\infty} f_\ell(r) : b_{r-\ell}^\dagger b_r : \quad .$$

In this section we shall exhibit a very natural basis of operators \mathcal{O}_ℓ and $\overline{\mathcal{O}}_\ell$ which satisfy the infinite dimensional algebra known as $W_{1+\infty}$ [12, 13]. In practice, this simply amounts to recognize a special basis of polynomials for the functions $f_\ell(r)$. However, this is not merely a change of basis. Indeed, by displaying the $W_{1+\infty}$ structure of the theory one can take advantage of the fact that this algebra can be realized also by bosonic operators. This means in particular, that once the algebraic content of the fermionic theory has been established, other realizations of the *same* algebra can be constructed in the bosonic language, and these can be chosen to diagonalize the total

hamiltonian. For this reason, we shall call this procedure *algebraic bosonization*. Before proceeding to rewrite the results of the previous section in terms of this new basis, we shall briefly review the essentials of a generic theory based on the $W_{1+\infty}$ algebra.

The $W_{1+\infty}$ algebra

The low-energy dynamics of simple $(1+1)$ -dimensional fermionic systems can be described by the (small) fluctuations of the zero-dimensional Fermi “surface” (which actually consists of an even number of Fermi points) around the one-dimensional Fermi sea. The effective degrees of freedom describing these fluctuations are the bosonized variables of the underlying fermionic theory (for formulations of bosonization that are close to our ideas see, *e.g.*, Refs. [5, 6, 7]). A systematic way of studying these fluctuations is to recognize their characteristic dynamical symmetry and organize them into irreducible representations of it. In Ref. [26] this dynamical symmetry has been identified as the infinite dimensional $W_{1+\infty}$ algebra [12]. More precisely, for those systems in which parity is unbroken, for example in the Heisenberg and the Calogero-Sutherland models, the dynamical symmetry is given by the algebra $W_{1+\infty} \times \overline{W}_{1+\infty}$, where each factor is associated to each Fermi point. Specifically, $W_{1+\infty}$ is the chiral symmetry algebra of the right Fermi point whereas $\overline{W}_{1+\infty}$ is the antichiral symmetry algebra of the left Fermi point. In most of the discussion that follows in this Section, it will be enough to consider, without loss of generality, only one component, say the chiral one; however we will point out explicitly all cases in which the combination of the chiral and antichiral sectors is relevant.

In simple terms, the (chiral) $W_{1+\infty}$ symmetry is an extension of the usual (chiral) conformal symmetry of $(1+1)$ -dimensional relativistic systems with massless excitations. Here, the relativistic “massless” excitations are the small fluctuations of the fermions close to the Fermi points in the momentum space of the non-relativistic system. One can imagine that the extended conformal symmetry has its origin precisely in the corrections to the approximate linear dispersion law around the Fermi points that the system exhibits. Moreover, as we shall see by explicit construction, the inclusion of the extra generators of the enhanced symmetry allows us to produce a systematic $1/N$ expansion in a very natural way.

The $W_{1+\infty}$ algebra is generated by an infinite set of (chiral) currents V_n^i , which are characterized by a (momentum) mode index $n \in \mathbf{Z}$ and an integer conformal spin $h = i + 1 \geq 1$. Roughly speaking, the geometrical meaning of the index i is associated to the type of “multipole” deformation the current V_n^i can induce on physical states. These currents satisfy the algebra [12],

$$[V_n^i, V_m^j] = (jn - im)V_{n+m}^{i+j-1} + q(i, j, n, m)V_{n+m}^{i+j-3} + \cdots + \delta^{ij}\delta_{n+m,0} c d(i, n) , \quad (3.1)$$

where the structure constants $q(i, j, n, m)$ and $d(i, n)$ are polynomial in their arguments, c is the central charge, and the dots denote a finite number of terms involving

the operators V_{n+m}^{i+j-2k} (the complete expression of Eq. (3.1) is a bit cumbersome and is given in the Appendix).

In particular, the operators V_n^0 satisfy the Abelian current algebra (Kac-Moody algebra) $\widehat{U(1)}$, while the operators V_n^1 close the Virasoro algebra [8], that is

$$[V_n^0, V_m^0] = c n \delta_{n+m,0} \quad , \quad (3.2)$$

$$[V_n^1, V_m^0] = -m V_{n+m}^0 \quad , \quad (3.3)$$

$$[V_n^1, V_m^1] = (n-m)V_{n+m}^1 + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad . \quad (3.4)$$

The eigenvalues of V_0^0 and V_0^1 are identified, respectively, as the charge and conformal dimension of a chiral excitation.

Other algebraic relations contained in Eq. (3.1), which will be useful in the next sections, are

$$\begin{aligned} [V_n^2, V_m^0] &= -2m V_{n+m}^1 \quad , \\ [V_n^2, V_m^1] &= (n-2m) V_{n+m}^2 - \frac{1}{6}(m^3-m) V_{n+m}^0 \quad , \\ [V_n^2, V_m^2] &= (2n-2m) V_{n+m}^3 + \frac{n-m}{15}(2n^2+2m^2-nm-8) V_{n+m}^1 \\ &\quad + c \frac{n(n^2-1)(n^2-4)}{180} \delta_{n+m,0} \quad . \end{aligned} \quad (3.5)$$

It is interesting to note that the operators V_0^i commute with each other for any i , and thus are the generators of the Cartan subalgebra of $W_{1+\infty}$.

In the classical limit, all terms but the first in the r.h.s. of Eq. (3.1) vanish; the resulting algebra is the classical algebra w_∞ of area-preserving diffeomorphisms, which can be understood as originating from all classical deformations of the density which conserve the number of particles [26].

A chiral $W_{1+\infty}$ theory is defined by a Hilbert space constructed out of a set of irreducible, unitary, highest-weight representations of the $W_{1+\infty}$ algebra, which are closed under the fusion rules for making composite states. If the parity symmetry is unbroken, the complete Hilbert space is obtained by combining chiral and antichiral representations of $W_{1+\infty}$. This is a simple extension of the well-known construction of conformal field theories. For this reason, and also because $W_{1+\infty}$ contains the Virasoro algebra as a subalgebra, a $W_{1+\infty}$ theory is called an extended conformal field theory.

All such theories can be completely classified thanks to the crucial work of Kac and Radul [13], in which all irreducible, unitary, quasi-finite highest-weight representations of (chiral) $W_{1+\infty}$ have been constructed. Such representations exist only

if the central charge is a positive integer, *i.e.* $c \in \mathbf{Z}_+$ ⁵. They are characterized by an c -dimensional weight vector \vec{Q} with real elements, and are built on top of a highest weight state $|\vec{Q}\rangle$, which satisfies

$$V_n^i |\vec{Q}\rangle = 0 \quad (3.6)$$

for any $n > 0$ and $i \geq 0$, and

$$V_0^i |\vec{Q}\rangle = \sum_{\alpha=1}^c m^i(Q_\alpha) |\vec{Q}\rangle \quad , \quad (3.7)$$

where $m^i(Q)$ are i -th order polynomials. In particular, for $i = 0, 1, 2$,

$$\begin{aligned} m^0(Q) &= Q \quad , \\ m^1(Q) &= \frac{1}{2} Q^2 \quad , \\ m^2(Q) &= \frac{1}{3} Q^3 \quad . \end{aligned} \quad (3.8)$$

The eigenvalue of V_0^0 is the charge (defined by the $\widehat{U(1)}$ symmetry of local particle number conservation) of $|\vec{Q}\rangle$, while that of V_0^1 gives its conformal weight.

The complete highest weight representation (the so-called Verma module) is obtained by constructing all the descendant states of $|\vec{Q}\rangle$. These correspond to neutrally charged (particle-hole) excitations above $|\vec{Q}\rangle$, and are defined as follows

$$|\vec{Q}, \{k_i\}\rangle = V_{-k_1}^0 V_{-k_2}^0 \cdots V_{-k_s}^0 |\vec{Q}\rangle \quad , \quad k_1 \geq k_2 \geq \cdots \geq k_s > 0 \quad . \quad (3.9)$$

The quantity $k = \sum_{i=1}^s k_i$ represents the total momentum of the excitation measured with respect to $|\vec{Q}\rangle$, and it is also known as the *level* of the descendant state $|\vec{Q}, \{k_i\}\rangle$.

We conclude this brief survey by pointing out that the effective theory of a Luttinger system [5, 2] is a $W_{1+\infty} \times \overline{W}_{1+\infty}$ conformal theory with $c = \bar{c} = 1$ [26]. In this case the highest weight vector \vec{Q} is one-dimensional and the corresponding highest weight states are denoted by $|Q\rangle$ in the right and $|\overline{Q}\rangle$ in the left sectors, with Q and \overline{Q} being their respective charges. Later on, we will show that this theory can be realized both by a free fermion and by a free compactified boson.

The Weyl fermion realization of $W_{1+\infty}$

In Section 2 we have seen that the effective hamiltonian of the $(1+1)$ -dimensional systems we considered, is constructed entirely in terms of the chiral and antichiral

⁵ For the special case of the Luttinger systems, like the Heisenberg and the Calogero-Sutherland models described in this paper, we have $c = 1$ [11, 26].

relativistic Weyl fermions (2.27) and (2.28). We now focus on one of them, say the chiral one, which is known to describe a $c = 1$ conformal theory. The Fermi sea for this theory is given by a highest-weight state $|\Omega\rangle$ satisfying

$$V_n^i |\Omega\rangle = 0 \quad (3.10)$$

for all $n \geq 0$ and $i \geq 0$.

The *neutrally charged* particle-hole excitations above the ground state are described by the descendant states of $|\Omega\rangle$ (see Eq. (3.9)). However, it is also natural to consider *charged excitations*, which manifest themselves as an excess or a defect of charge around the Fermi point. When considering the complete chiral-antichiral theory, one realizes that there are two physically inequivalent ways of producing these excitations: by addition or subtraction of extra particles with momentum close to the Fermi points, and by the coupling of the system to an external probe producing an overall shift in momentum which conserves the total particle number [11, 26]. The latter are the analogs of Laughlin's quasi-particle excitations in the quantum Hall effect [15], as seen from the edge of a sample. The analogy is evident because Laughlin's quantum incompressible fluids are configuration space analogs of a Fermi sea. In the algebraic formalism, these excitations appear as further highest-weight states, which have non-vanishing eigenvalues for all the V_0^i and \bar{V}_0^i .

Therefore, the Hilbert space of a Weyl fermion consists of an infinity of $c = 1$ $W_{1+\infty}$ representations, Eqs. (3.6) and (3.7), which are characterized by an *integer* weight representing the charge of the highest weight state⁶. Obviously, the Hilbert space of an antichiral fermion is isomorphic to that of a chiral one, and defines a $\bar{c} = 1$ $\bar{W}_{1+\infty}$ conformal field theory. Therefore, we are led to characterize the thermodynamic limit of a fermionic Fermi system in which parity is unbroken, as a $W_{1+\infty} \times \bar{W}_{1+\infty}$ theory.

In physical applications, operators and fields are naturally defined on a spatially compact space, like the circle of radius R of Eq. (2.29). The corresponding Minkowskian theory is then defined on the cylinder formed by the spatial circle times the real line that represents the time coordinate. On the other hand, in the mathematical literature operators and fields are conventionally defined on the complex plane. Therefore, it is convenient to map the physical operators from the cylinder to the plane, where one may use the mathematical results. There is a well-known conformal mapping between the cylinder ($u = \tau - iR\theta$), and the conformal plane (z), namely

$$z = \exp\left(\frac{u}{R}\right) = \exp\left(\frac{\tau}{R} - i\theta\right) \quad , \quad (3.11)$$

⁶ The more general representations with $c = m$ a positive integer can be obtained by considering m independent Weyl fermions (see [28] for details).

where τ denotes the euclidean time. Under this map, the Weyl fermion (2.27), which is a primary field of weight $h = 1/2$, takes the form (at $\tau = 0$)

$$\begin{aligned} F(z) &= \left(\frac{du}{dz} \right)^{1/2} F_+(\theta) = \sum_{r=-\infty}^{\infty} e^{ir\theta} a_r \quad , \\ F^\dagger(z) &= \left(\frac{du}{dz} \right)^{1/2} F_+^\dagger(\theta) = \sum_{r=-\infty}^{\infty} e^{-i(r-1)\theta} a_r^\dagger \quad . \end{aligned} \quad (3.12)$$

Notice the well-known fact that due to the map (3.11), the definition of $F^\dagger(z)$ differs from the naive expression. The expression for the antichiral fields can be simply obtained from Eq. (3.12) by replacing a_r with b_r , and θ with $-\theta$, *i.e.* z with \bar{z} .

The representation of the $W_{1+\infty}$ generators as operators acting on the Hilbert space of a Weyl fermion is discussed in detail in the Appendix. Here we simply recall that it is obtained by sandwiching specific polynomials $g_n^i = (-1)^{i+1} f_n^i$ in $D \equiv z\partial$ (with f_n^i given in the Appendix) between the field operators $F(z)$ and $F^\dagger(z)$, according to

$$V_n^i = \oint \frac{dz}{2\pi i} : F^\dagger(z) z^n g_n^i(D) F(z) : \quad . \quad (3.13)$$

Here the integration is carried clockwise over the unit circle, and the normal ordering $: \quad :$ is defined canonically with respect to the ground state $|\Omega\rangle$ as in Section 2. Since the anticommutator of F and F^\dagger is a delta function in Fock space, the operators V_n^i defined above clearly represent the $W_{1+\infty}$ algebra as long as the functions f_n^i do so. One can also verify that the zero modes V_0^i defined in this way have the eigenvalues (3.8) when acting on fermion states of charge Q . Furthermore, Eq. (3.13) shows how these operators are written in the canonical form $(F^\dagger F)$ of quantum field theory.

Using the explicit formulae for f_n^i given in the Appendix, we obtain the Fock space expressions of the generators for the chiral sector (of course, analogous formulae hold for the antichiral sector replacing a_r with b_r). For the first few values of the conformal spin, these are

$$\begin{aligned} V_n^0 &= \sum_{r=-\infty}^{\infty} : a_{r-n}^\dagger a_r : \quad , \\ V_n^1 &= \sum_{r=-\infty}^{\infty} \left(r - \frac{n+1}{2} \right) : a_{r-n}^\dagger a_r : \quad , \\ V_n^2 &= \sum_{r=-\infty}^{\infty} \left(r^2 - (n+1)r + \frac{(n+1)(n+2)}{6} \right) : a_{r-n}^\dagger a_r : \quad , \\ V_n^3 &= \sum_{r=-\infty}^{\infty} \left(r^3 - \frac{3}{2}(n+1)r^2 + \frac{6n^2+15n+11}{10}r \right. \\ &\quad \left. - \frac{(n+1)(n+2)(n+3)}{20} \right) : a_{r-n}^\dagger a_r : \quad . \end{aligned} \quad (3.14)$$

The corresponding currents on the complex plane, *i.e.*

$$V^i(z) \equiv \sum_{n=-\infty}^{\infty} V_n^i z^{-n-i-1} \quad ,$$

can then be written as follows

$$\begin{aligned} V^0(z) &= :F^\dagger F: \quad , \\ V^1(z) &= \frac{1}{2} : \partial (F^\dagger F) : - : F^\dagger \partial F : \quad , \\ V^2(z) &= \frac{1}{6} : \partial^2 (F^\dagger F) : - : \partial F^\dagger \partial F : \quad , \\ V^3(z) &= \frac{1}{24} : \partial^3 (F^\dagger F) : - \frac{1}{2} (: \partial F^\dagger \partial^2 F : - : \partial^2 F^\dagger \partial F :) \quad . \end{aligned} \tag{3.15}$$

These expressions will be extensively used in the next sections.

The boson realization of $W_{1+\infty}$

There is yet another realization of the $c = 1$ $W_{1+\infty}$ algebra which will be useful for our purposes. It arises through the abelian bosonization of the fermionic fields $F(z)$ and $F^\dagger(z)$ introduced in the previous paragraph. To see this let us consider a chiral boson field $\varphi(z)$ defined on the complex plane such that

$$F(z) \equiv : \exp(i\varphi(z)) : \quad , \quad F^\dagger(z) \equiv : \exp(-i\varphi(z)) : \quad .$$

Then one can show that the operator $\partial\varphi(z)$ is a chiral current of conformal weight $(1,0)$, which is identified with the lowest spin $W_{1+\infty}$ generator $V^0(z)$. The Fourier expansion of this operator is given by

$$\partial\varphi(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n-1} \quad , \tag{3.16}$$

with $\alpha_n \equiv V_n^0$. From the commutation relations $[\alpha_n, \alpha_m] = n \delta_{m+n,0}$ (see Eq. (3.2)), it follows that the oscillators α_n with $n > 0$ ($n < 0$) are destruction (creation) operators, and their normal ordering is canonically defined.

The higher spin $W_{1+\infty}$ generators can be constructed in the bosonic language out of the current (3.16) through a generalized Sugawara construction [13]. For example, the lowest spin generators are given by

$$\begin{aligned} V^0(z) &= \partial\varphi(z) \quad , \\ V^1(z) &= \frac{1}{2} : (\partial\varphi(z))^2 : \quad , \\ V^2(z) &= \frac{1}{3} : (\partial\varphi(z))^3 : \quad , \\ V^3(z) &= \frac{1}{4} : (\partial\varphi(z))^4 : - \frac{3}{20} : (\partial^2\varphi(z))^2 : + \frac{1}{10} : \partial\varphi(z) \partial^3\varphi(z) : \quad , \end{aligned} \tag{3.17}$$

and their corresponding modes are

$$\begin{aligned}
V_n^0 &= \alpha_n \quad , \\
V_n^1 &= \frac{1}{2} \sum_{\ell=-\infty}^{\infty} : \alpha_\ell \alpha_{n-\ell} : \quad , \\
V_n^2 &= \frac{1}{3} \sum_{\ell,k=-\infty}^{\infty} : \alpha_\ell \alpha_k \alpha_{n-\ell-k} : \quad , \\
V_n^3 &= \frac{1}{4} \sum_{\ell,k,m=-\infty}^{\infty} : \alpha_\ell \alpha_k \alpha_m \alpha_{n-\ell-k-m} : \\
&\quad + \frac{1}{20} \sum_{\ell=-\infty}^{\infty} (2n - 5\ell + 1) (n - \ell + 1) : \alpha_\ell \alpha_{n-\ell} : \quad .
\end{aligned} \tag{3.18}$$

As we have stressed above, the effective theory of physically interesting systems (like the Luttinger model) is a $W_{1+\infty} \times \overline{W}_{1+\infty}$ conformal theory with $c = \bar{c} = 1$. In the bosonic language, it can be realized by a free compactified field Φ [8] with euclidean action

$$S = \frac{1}{2\pi r} \int d^2z \, \partial\Phi \bar{\partial}\Phi \quad , \tag{3.19}$$

and a compactification radius r defined by $\Phi \equiv \Phi + 2\pi r$. The equations of motion for Φ give

$$\Phi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}) \quad ,$$

where $\varphi(z)$ and $\bar{\varphi}(\bar{z})$ are the chiral and antichiral components of the boson field Φ , respectively. The spectrum of this theory is well-known [8] and can be given in terms of two numbers m and n (with $m \in \mathbf{Z}$ and $(m/2 - n) \in \mathbf{Z}$). In particular, the highest weight states denoted by $|m, n\rangle$, are such that

$$\begin{aligned}
V_0^0 |m, n\rangle &= Q |m, n\rangle \quad , \\
\overline{V}_0^0 |m, n\rangle &= \overline{Q} |m, n\rangle \quad ,
\end{aligned} \tag{3.20}$$

where the right and left charges Q and \overline{Q} are respectively

$$Q = \frac{m}{2r} + r n \quad , \quad \overline{Q} = \frac{m}{2r} - r n \quad . \tag{3.21}$$

Note that the two chiral components of Φ are not totally independent since there is an overall constraint of “charge conservation”. Furthermore, for general values of r , the charges Q and \overline{Q} are not necessarily integers, as opposed to the fermionic case. Indeed, it is well-known [8] that the free fermion representation discussed in the previous paragraph is equivalent to that of a compactified bosonic field only for $r = 1$. However, in the bosonic realization, one can freely change the value of the compactification radius without changing the algebra. It is precisely this freedom which will be exploited in the following sections to diagonalize the effective hamiltonians of the Calogero-Sutherland and the Heisenberg models.

4 The $W_{1+\infty}$ structure of the Calogero-Sutherland model

In this section we demonstrate that the $W_{1+\infty}$ algebra is the natural framework to interpret and understand the results of Section 2. In particular we show that the effective hamiltonian of the Calogero-Sutherland model, Eq. (2.61), can be nicely written in terms of $W_{1+\infty}$ currents, and that the low-energy spectrum of the model follows directly from the $W_{1+\infty}$ representation theory. To this aim, we introduce *two* sets of $W_{1+\infty}$ generators, one for each Fermi point: the right ones, denoted by V_n^i , are bilinear forms in the a oscillators according to Eq. (3.14), whilst the left ones, denoted by \bar{V}_n^i , are bilinear in the b oscillators. Then, it is easy to see that Eqs. (2.62)-(2.64) become

$$\mathcal{H}_{(0)} = \frac{1}{4}(1+g) \left(V_0^0 + \bar{V}_0^0 \right) \quad , \quad (4.1)$$

$$\mathcal{H}_{(1)} = \left(1 + \frac{g}{2} \right) \left(V_0^1 + \bar{V}_0^1 \right) + \frac{g}{2} \sum_{\ell=-\infty}^{\infty} V_\ell^0 \bar{V}_\ell^0 \quad , \quad (4.2)$$

$$\begin{aligned} \mathcal{H}_{(2)} = & \left(1 + \frac{g}{4} \right) \left(V_0^2 + \bar{V}_0^2 \right) - \frac{1}{12}(1+g) \left(V_0^0 + \bar{V}_0^0 \right) \\ & - \frac{g}{4} \sum_{\ell=-\infty}^{\infty} |\ell| \left(V_\ell^0 V_{-\ell}^0 + \bar{V}_{-\ell}^0 \bar{V}_\ell^0 + 2 V_\ell^0 \bar{V}_\ell^0 \right) \\ & + \frac{g}{2} \sum_{\ell=-\infty}^{\infty} \left(V_\ell^1 \bar{V}_\ell^0 + V_\ell^0 \bar{V}_\ell^1 \right) \quad . \end{aligned} \quad (4.3)$$

As explained in Section 3, the operators V_ℓ^0 and \bar{V}_ℓ^0 satisfy a $\widehat{U(1)}$ Kac-Moody algebra with central charge $c = 1$ (see Eq. (3.2)), and can be identified with the right and left modes of a non-chiral bosonic field compactified on a circle, respectively. The highest weight states of this Kac-Moody algebra are labeled by two quantum numbers ΔN and ΔD , and will be denoted by $|\Delta N, \Delta D\rangle_0$. The meaning of these states is particularly clear in the original fermionic description. In fact, $|\Delta N, \Delta D\rangle_0$ is obtained from the fermionic ground state $|\Omega\rangle$ by adding ΔN particles, and by moving ΔD particles from the right to the left Fermi point. To be precise, given any two integer numbers q and \bar{q} , we have

$$|\Delta N, \Delta D\rangle_0 = A_q B_{\bar{q}} |\Omega\rangle \quad ,$$

where

$$\Delta N = q + \bar{q} \quad , \quad \Delta D = \frac{q - \bar{q}}{2} \quad ,$$

and

$$A_q = \begin{cases} a_1^\dagger a_2^\dagger \cdots a_q^\dagger & \text{for } q = 1, 2, \dots \\ 1 & \text{for } q = 0 \\ a_0 a_{-1} \cdots a_{q+1} & \text{for } q = -1, -2, \dots \end{cases}$$

and analogously for $B_{\bar{q}}$ with the b oscillators. The descendant states, denoted by $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0$, have also a simple interpretation in the original fermionic description: in fact, they coincide with the particle-hole excitations obtained from $|\Delta N, \Delta D\rangle_0$ by acting with operators of the kind $a_r^\dagger a_s$, $b_r^\dagger b_{\bar{s}}$ or combinations thereof.

Using the explicit expressions of V_0^0 and \bar{V}_0^0 given in Eq. (3.14), one can easily check that

$$\begin{aligned} V_0^0 |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} + \Delta D \right) |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0, \\ \bar{V}_0^0 |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} - \Delta D \right) |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0. \end{aligned} \quad (4.4)$$

Comparing these eigenvalues with Eq. (3.21), we deduce that V_n^0 and \bar{V}_n^0 are the modes of a bosonic field compactified on a circle of radius $r_0 = 1$. This field describes the density fluctuations of the original free fermionic fields of Eqs. (2.27) and (2.28).

Let us now consider the $1/N$ -term of the effective hamiltonian given by Eq. (4.2). Due to the left-right mixed part proportional to g , $\mathcal{H}_{(1)}$ is not diagonal on the states $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_0$ previously considered. However, it is not difficult to diagonalize it. To do so, we use the Sugawara construction (see Eq. (3.18)), and replace $(V_0^1 + \bar{V}_0^1)$ with a quadratic form in V_ℓ^0 and \bar{V}_ℓ^0 , so that $\mathcal{H}_{(1)}$ becomes

$$\begin{aligned} \mathcal{H}_{(1)} &= \frac{1}{2} \left(1 + \frac{g}{2} \right) \left[(V_0^0)^2 + (\bar{V}_0^0)^2 \right] + \frac{g}{2} V_0^0 \bar{V}_0^0 \\ &+ \sum_{\ell=1}^{\infty} \left[\left(1 + \frac{g}{2} \right) (V_{-\ell}^0 V_\ell^0 + \bar{V}_{-\ell}^0 \bar{V}_\ell^0) + \frac{g}{2} (V_\ell^0 \bar{V}_\ell^0 + V_{-\ell}^0 \bar{V}_{-\ell}^0) \right]. \end{aligned} \quad (4.5)$$

This expression exhibits the essential feature of the algebraic bosonization: through the Sugawara construction, a two-fermion term has been replaced with a two-boson term satisfying the same algebraic properties [19]. The quadratic form in the r.h.s. of Eq. (4.5) can now be diagonalized by means of the following Bogoliubov transformation

$$\begin{aligned} W_\ell^0 &= V_\ell^0 \cosh \beta + \bar{V}_{-\ell}^0 \sinh \beta, \\ \bar{W}_\ell^0 &= V_{-\ell}^0 \sinh \beta + \bar{V}_\ell^0 \cosh \beta \end{aligned} \quad (4.6)$$

for all ℓ , with

$$\tanh 2\beta = \frac{g}{2+g}. \quad (4.7)$$

In fact, using Eq. (4.6) into Eq. (4.5), up to an irrelevant additive constant we get

$$\mathcal{H}_{(1)} = \frac{\lambda}{2} \left[(W_0^0)^2 + (\bar{W}_0^0)^2 \right] + \lambda \sum_{\ell=1}^{\infty} (W_{-\ell}^0 W_\ell^0 + \bar{W}_{-\ell}^0 \bar{W}_\ell^0), \quad (4.8)$$

where

$$\lambda \equiv \exp(2\beta) = \sqrt{1+g} \quad . \quad (4.9)$$

In writing Eq. (4.8) we have used the property that W_ℓ^0 and \overline{W}_ℓ^0 satisfy an abelian Kac-Moody algebra with central charge $c = 1$ like the original operators V_ℓ^0 and \overline{V}_ℓ^0 (cf. Eq. (3.2)). By means of the generalized Sugawara construction, we can then define a new realization of the $W_{1+\infty}$ algebra whose generators W_n^i and \overline{W}_n^i are forms of degree $(i+1)$ in W_ℓ^0 and \overline{W}_ℓ^0 respectively, like those of Eq. (3.18). Consequently, we can rewrite Eq. (4.8) as follows

$$\mathcal{H}_{(1)} = \lambda \left(W_0^1 + \overline{W}_0^1 \right) \quad , \quad (4.10)$$

while $\mathcal{H}_{(0)}$, given in Eq. (4.1), simply becomes

$$\mathcal{H}_{(0)} = \frac{\sqrt{\lambda^3}}{4} \left(W_0^0 + \overline{W}_0^0 \right) \quad . \quad (4.11)$$

The effective hamiltonian of the Calogero-Sutherland model up to order $1/N$, *i.e.*

$$\begin{aligned} \mathcal{H}_{(1/N)} &\equiv (2\pi\rho_0)^2 \left(\mathcal{H}_{(0)} + \frac{1}{N} \mathcal{H}_{(1)} \right) \\ &= (2\pi\rho_0\sqrt{\lambda})^2 \left[\left(\frac{\sqrt{\lambda}}{4} W_0^0 + \frac{1}{N} W_0^1 \right) + (W \leftrightarrow \overline{W}) \right] \quad , \end{aligned} \quad (4.12)$$

exhibits a left-right factorization in the *new* realization of the $W_{1+\infty}$ algebra. In particular, in the r.h.s. of Eq. (4.12) we recognize the typical structure of the hamiltonian of a conformal field theory, whose spectrum is known.

Notice that $\mathcal{H}_{(1/N)}$ is not diagonal on $|\Delta N, \Delta D; \{k_i\}, \{\overline{k}_j\}\rangle_0$, because the highest weight states of the new algebra do not coincide with the vectors $|\Delta N, \Delta D\rangle_0$, as is clear from Eq. (4.6). However, since the new charge operators, W_0^0 and \overline{W}_0^0 , depend only on V_0^0 and \overline{V}_0^0 , the Bogoliubov transformation does not mix states belonging to different Verma moduli. This implies that the new highest weight vectors are still characterized by the numbers ΔN and ΔD with the same meaning as before, but their charges are different. More precisely, the new highest weight states are defined by

$$W_\ell^i |\Delta N; \Delta D\rangle_W = \overline{W}_\ell^i |\Delta N; \Delta D\rangle_W = 0$$

for all $\ell > 0$ and $i \geq 0$, and

$$\begin{aligned} W_0^0 |\Delta N; \Delta D\rangle_W &= \left(\sqrt{\lambda} \frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\lambda}} \right) |\Delta N; \Delta D\rangle_W \\ \overline{W}_0^0 |\Delta N; \Delta D\rangle_W &= \left(\sqrt{\lambda} \frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\lambda}} \right) |\Delta N; \Delta D\rangle_W \quad . \end{aligned} \quad (4.13)$$

Comparing these last two equations with Eq. (3.21), we can deduce that W_ℓ^0 and \overline{W}_ℓ^0 are the modes of a bosonic field compactified on a circle of radius $r = 1/\sqrt{\lambda} =$

$\exp(-\beta)$. This field describes the density fluctuations of the *interacting* fermions of the Calogero-Sutherland model.

The highest weight states $|\Delta N, \Delta D\rangle_W$ together with their descendants, denoted by $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W$, form a *new* bosonic basis for our theory that has no simple expression in terms of the original free fermionic degrees of freedom. In fact, as is well-known, the Bogoliubov transformation, Eq. (4.6), is non-local in the fermionic operators. The main property of this new basis is that it diagonalizes the effective hamiltonian of the Calogero-Sutherland model up to order $1/N$. In fact, using Eqs. (3.8), (4.12) and (4.13), it is easy to check that

$$\mathcal{H}_{(1/N)} |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W = \mathcal{E}_{(1/N)} |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W \quad (4.14)$$

where

$$\mathcal{E}_{(1/N)} = \left(2\pi\rho_0\sqrt{\lambda}\right)^2 \left[\frac{\lambda}{4} \Delta N + \frac{1}{N} \left(\lambda \frac{(\Delta N)^2}{4} + \frac{(\Delta D)^2}{\lambda} + k + \bar{k} \right) \right] , \quad (4.15)$$

with $k = \sum_i k_i$ and $\bar{k} = \sum_j \bar{k}_j$. These eigenvalues are clearly degenerate when $k \geq 2$ or $\bar{k} \geq 2$. Notice that Eq. (4.15) can be written also as follows

$$\mathcal{E}_{(1/N)} = \mu \Delta N + \frac{2\pi v}{L} \left(\lambda \frac{(\Delta N)^2}{4} + \frac{(\Delta D)^2}{\lambda} + k + \bar{k} \right) ,$$

where μ is the chemical potential and v the Fermi velocity. Examining the structure of the energy eigenvalues (4.15) and comparing with those at $g = 0$, one can say that, up to order $1/N$, the Calogero-Sutherland interaction induces the following three effects:

1. a rescaling of the chemical potential

$$\mu_0 = \frac{(2\pi\rho_0)^2}{4} \longrightarrow \mu = \lambda^2 \mu_0 , \quad (4.16)$$

2. a rescaling of the Fermi velocity of the particles

$$v_0 = 2\pi\rho_0 \longrightarrow v = \lambda v_0 , \quad (4.17)$$

3. a change in the compactification radius of the bosonic field describing the fermion density fluctuations

$$r_0 = 1 \longrightarrow r = \frac{r_0}{\sqrt{\lambda}} . \quad (4.18)$$

It is interesting to observe that the rescalings in Eqs. (4.16)-(4.18) have their origin in the backward scattering processes of the Calogero-Sutherland model, which, as we have remarked at the end of Section 2, are the only interactions that contribute to the effective hamiltonian to order $1/N$. In particular, to lowest order in g , the change in the compactification radius is induced by the left-right mixed terms of $\mathcal{H}_{(1)}$. These are indeed the $1/N$ -terms of the backscattering hamiltonian (2.59), written in the bosonized language. Such terms have the generic form $V_\ell^0 \bar{V}_\ell^0$, and have conformal dimension $(1,1)$. Therefore, they are marginal operators, which cannot destroy the conformal symmetry of the free theory, but only change the realization of the conformal algebra [8]. In fact, these terms can be regarded as a marginal perturbation to Eq. (3.19) which drives the theory out from the free realization in terms of V_ℓ^0 and \bar{V}_ℓ^0 to the interacting realization in terms of W_ℓ^0 and \bar{W}_ℓ^0 . In this flow, the central charge of the conformal algebra remains unchanged while the compactification radius of the bosonic field varies according to Eq. (4.18).

The rescalings of the chemical potential and the Fermi velocity, Eqs. (4.16) and (4.17), have instead a different interpretation. In fact, to lowest order in g , they are produced by the left and right diagonal terms proportional to g in $\mathcal{H}_{(0)}$ and $\mathcal{H}_{(1)}$. If we trace back their origin, we see that these terms arise from the two-body part of the backscattering hamiltonian (2.60). Therefore, they are a normal ordering effect.

It is remarkable that despite their different origins, the rescalings in Eqs. (4.16)-(4.18) are characterized by only *one* function of the coupling constant, namely the parameter λ defined in Eq. (4.9). This fact implies that they are not independent from one another; for example, one has

$$v_0 r_0^2 = v r^2 \quad . \quad (4.19)$$

This relation is typical of the Luttinger model [27], and actually holds true for all systems whose hamiltonian at order $1/N$ has the same form as in Eqs. (4.5) or (4.10), that is, in all cases for which the interaction can be simply taken into account by means of a Bogoliubov transformation like Eq. (4.6). However, not all models fit into this category.

At this point a few comments are in order. We should keep in mind that the derivation of the effective theory, as presented in Section 2, is strictly perturbative; thus, in all previous formulas, we should always understand a perturbative expansion in the coupling constant g , and keep only the first order corrections. However, if we limit our analysis to the $1/N$ -terms, nothing prevents us from improving our results and extend them to all orders in g . Indeed, when we perform the Bogoliubov transformation (4.6), we diagonalize the hamiltonian $\mathcal{H}_{(1)}$ *exactly*, and the resulting expression depends on the coupling constant only through λ , which contains all powers of g (see Eq. (4.9))! This improvement is a well-known fact in the Luttinger model [27], but we would like to stress that in our case it can be done only if we disregard

the $O(1/N^2)$ -terms of the hamiltonian. In fact, as we shall see momentarily, the Bogoliubov transformation (4.6) does not diagonalize $\mathcal{H}_{(2)}$.

To investigate this issue, let us analyze the $1/N^2$ -term of the effective hamiltonian given by Eq. (4.3). Using the generalized Sugawara construction (3.18), we first rewrite $\mathcal{H}_{(2)}$ as a cubic form in V_ℓ^0 and \bar{V}_ℓ^0 , and then perform the Bogoliubov transformation (4.6) in order to express it in terms of the new generators of the $W_{1+\infty}$ algebra. A straightforward calculation leads to

$$\mathcal{H}_{(2)} = \mathcal{H}'_{(2)} + \mathcal{H}''_{(2)}$$

where

$$\begin{aligned} \mathcal{H}'_{(2)} = & \sqrt{\lambda} (W_0^2 + \bar{W}_0^2) - \frac{\sqrt{\lambda^3}}{12} (W_0^0 + \bar{W}_0^0) \\ & - \frac{g}{2\lambda} \sum_{\ell=1}^{\infty} \ell (W_{-\ell}^0 W_\ell^0 + \bar{W}_{-\ell}^0 \bar{W}_\ell^0) \quad , \end{aligned} \quad (4.20)$$

and

$$\mathcal{H}''_{(2)} = -\frac{g}{2\lambda} \sum_{\ell=1}^{\infty} \ell (W_\ell^0 \bar{W}_\ell^0 + W_{-\ell}^0 \bar{W}_{-\ell}^0) \quad . \quad (4.21)$$

Neither $\mathcal{H}'_{(2)}$ nor $\mathcal{H}''_{(2)}$ are diagonal in the basis $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W$ considered so far. In general, these states are not eigenstates of $(W_0^2 + \bar{W}_0^2)$, and hence cannot be eigenstates of $\mathcal{H}'_{(2)}$; moreover, since they have definite values of k and \bar{k} , they cannot be eigenstates of $\mathcal{H}''_{(2)}$ either, because this operator mixes the left and right sectors.

It is not difficult, however, to overcome these problems. Since $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$ commute with each other, it is always possible to find suitable combinations of the states $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W$ with fixed k and \bar{k} that diagonalize simultaneously $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$ (see below for a few explicit examples). Notice that by diagonalizing also $\mathcal{H}'_{(2)}$, we lift the degeneracy of the spectrum that appeared to order $1/N$. We denote the space of the eigenstates of $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$ by $\mathcal{W}(\Delta N, \Delta D)$.

The term $\mathcal{H}''_{(2)}$ instead can be treated perturbatively, but only to first order in g ; in fact at higher orders, also the spurious states introduced when sending the momentum cutoff $\Lambda_0 \rightarrow \infty$ (see the discussion before Eq. (2.27) in Section 2) would contribute as intermediate states. These contributions, however, would be meaningless because the hamiltonian to order $O(1/N^2)$ is not even bounded below. From Eq. (4.21) it is easy to check that $\mathcal{H}''_{(2)}$ has vanishing expectation value on any state belonging to $\mathcal{W}(\Delta N, \Delta D)$, and thus, according to (non-degenerate) perturbation theory, $\mathcal{H}''_{(2)}$ has no effect on the energy spectrum to first order in g .

In view of these considerations, we can neglect $\mathcal{H}''_{(2)}$ and regard as the effective hamiltonian of the Calogero-Sutherland model the following operator

$$\mathcal{H}_{CS} \equiv \mathcal{H}_{(1/N)} + (2\pi\rho_0)^2 \frac{1}{N^2} \mathcal{H}'_{(2)}$$

$$\begin{aligned}
&= \left(2\pi\rho_0\sqrt{\lambda}\right)^2 \left\{ \left[\frac{\sqrt{\lambda}}{4} W_0^0 + \frac{1}{N} W_0^1 + \frac{1}{N^2} \left(\frac{1}{\sqrt{\lambda}} W_0^2 - \frac{\sqrt{\lambda}}{12} W_0^0 \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{g}{2\lambda^2} \sum_{\ell=1}^{\infty} \ell W_{-\ell}^0 W_{\ell}^0 \right) \right] + (W \leftrightarrow \overline{W}) \right\} \quad , \quad (4.22)
\end{aligned}$$

where λ is defined in Eq. (4.9). Obviously, to be consistent with our perturbative approach, in the r.h.s. of Eq. (4.22) we should keep only terms that are linear in g .

It is now interesting to compare the eigenvalues of \mathcal{H}_{CS} with the exact low-energy spectrum of the Calogero-Sutherland model obtained from the Bethe Ansatz solution [23, 11]. It is known that the energy (with respect to the ground state) of any configuration of the system can be written as

$$\tilde{\mathcal{E}} = \sum_{j=1}^N p_j^2 - E_0 \quad (4.23)$$

where E_0 is the ground state energy and p_j are the pseudomomenta of the particles which satisfy the Bethe Ansatz equations. In our case these take a particularly simple form; in fact they are

$$L p_j = 2\pi I_j + \pi (\xi - 1) \sum_{\ell=1}^N \text{sgn}(p_j - p_{\ell}) \quad , \quad (4.24)$$

where

$$\xi = \frac{1 + \sqrt{1 + 2g}}{2} \quad , \quad (4.25)$$

and I_j are the integer quantum numbers that specify the levels occupied by the particles. For example, the ground state is characterized by the set

$$\{I_j^0\} = \{-n_F, -n_F + 1, \dots, n_F\} \quad .$$

with $n_F = (N - 1)/2$.

A low-lying excitation above the ground state can be obtained in three different ways: by adding ΔN particles to the system ($\Delta N \in \mathbf{Z}$), by moving ΔD particles from the left to the right Fermi point ($\Delta N/2 - \Delta D \in \mathbf{Z}$), or by creating particle-hole pairs at levels n_j and \bar{n}_j on the right and on the left respectively. Therefore, a generic low-lying excitation is labeled by the following quantum numbers

$$I_j = \tilde{I}_j^0 + \Delta D - \bar{n}_j + n_{N'-j+1} \quad (4.26)$$

where $j = 1, 2, \dots, N'$, and

$$\{\tilde{I}_j^0\} = \left\{ -\frac{N'-1}{2}, -\frac{N'-1}{2} + 1, \dots, \frac{N'-1}{2} \right\}$$

with

$$N' = N + \Delta N \quad .$$

Moreover, the integer numbers n_j are ordered according to

$$n_1 \geq n_2 \geq \dots \geq 0$$

and are different from zero only if $j < N$ (and analogously for the \bar{n}_j).

By using Eqs. (4.23) and (4.24) and generalizing to order $1/N^2$ the procedure presented in Ref. [11], we can easily derive the exact energy of the excitation described by the numbers (4.26); this is

$$\begin{aligned} \tilde{\mathcal{E}} = & \left(2\pi\rho_0\sqrt{\xi}\right)^2 \left\{ \left[\frac{\sqrt{\xi}}{4} Q + \frac{1}{N} \left(\frac{1}{2} Q^2 + n \right) + \frac{1}{N^2} \left(\frac{1}{3\sqrt{\xi}} Q^3 - \frac{\sqrt{\xi}}{12} Q \right. \right. \right. \\ & \left. \left. + \frac{2n}{\sqrt{\xi}} Q + \frac{\sum_j n_j^2}{\xi} - \sum_j (2j-1) n_j \right) \right] + (Q \leftrightarrow \bar{Q}, \{n_j\} \leftrightarrow \{\bar{n}_j\}) \right\} \quad , \end{aligned} \quad (4.27)$$

where

$$n = \sum_j n_j \quad , \quad \bar{n} = \sum_j \bar{n}_j$$

and

$$Q = \sqrt{\xi} \frac{\Delta N}{2} + \frac{\Delta D}{\sqrt{\xi}} \quad , \quad \bar{Q} = \sqrt{\xi} \frac{\Delta N}{2} - \frac{\Delta D}{\sqrt{\xi}} \quad , \quad (4.28)$$

Of course, being an exact result, Eq. (4.27) holds to all orders in g . Comparing Eqs. (4.9) and (4.25), we see that

$$\xi = \lambda + O(g^2) \quad . \quad (4.29)$$

Thus, to first order in g , Q and \bar{Q} of Eq. (4.28) coincide with the eigenvalues of W_0^0 and \bar{W}_0^0 given in Eq. (4.13); conversely, these latter can be interpreted as the first-order approximation to the exact ones. Notice that Q and \bar{Q} have the structure of the zero mode charges of a non-chiral bosonic field compactified on a circle of radius

$$\tilde{r} = \frac{1}{\sqrt{\xi}} \quad . \quad (4.30)$$

Indeed, this is the exact value of the compactification radius of the bosonic field describing the density fluctuations of the fermions in the Calogero-Sutherland model [11]. From Eq. (4.27) we can also see that the exact value of the chemical potential is

$$\tilde{\mu} = \frac{(2\pi\rho_0)^2}{4} \xi^2 \quad , \quad (4.31)$$

while the exact Fermi velocity is

$$\tilde{v} = 2\pi\rho_0 \xi \quad . \quad (4.32)$$

These expressions are similar to those in Eqs. (4.16)-(4.18) with ξ in place of λ . Of course, due to Eq. (4.29), \tilde{r} , $\tilde{\mu}$ and \tilde{v} coincide, respectively, with r , μ and v , to first order in g . It is worthwhile pointing out that all low-energy effects of the Calogero-Sutherland interaction are encoded entirely in a unique quantity, namely the parameter ξ , which in the Bethe Ansatz literature is known as dressed charge factor [4].

Since the exact results can be obtained from the perturbative ones simply by changing λ into ξ , we are led to conjecture that the *exact* effective hamiltonian of the Calogero-Sutherland model is given by Eq. (4.22) with ξ , defined in Eq. (4.25), in place of λ , that is

$$\begin{aligned} \tilde{\mathcal{H}}_{CS} = & \left(2\pi\rho_0\sqrt{\xi}\right)^2 \left\{ \left[\frac{\sqrt{\xi}}{4} W_0^0 + \frac{1}{N} W_0^1 + \frac{1}{N^2} \left(\frac{1}{\sqrt{\xi}} W_0^2 - \frac{\sqrt{\xi}}{12} W_0^0 \right. \right. \right. \\ & \left. \left. \left. - \frac{g}{2\xi^2} \sum_{\ell=1}^{\infty} \ell W_{-\ell}^0 W_{\ell}^0 \right) \right] + (W \leftrightarrow \overline{W}) \right\} . \end{aligned} \quad (4.33)$$

We may consider this operator as a non-perturbative improvement of \mathcal{H}_{CS} which was derived in perturbation theory.

Evidence for the validity of our conjecture, which is certainly true to order $1/N$ (see [11]), is provided by the calculation of the eigenvalues of $\tilde{\mathcal{H}}_{CS}$. We will check on same explicit examples that these eigenvalues coincide with the exact energy of the low-lying excitations given in Eq. (4.27). To this aim, let us first consider the highest weight state $|\Delta N, \Delta D\rangle_W$ that satisfies Eq. (4.13) with ξ in place of λ . Using Eqs. (3.7) and (3.8) for $c = 1$, it is immediate to see that $|\Delta N, \Delta D\rangle_W$ is an eigenstate of $\tilde{\mathcal{H}}_{CS}$ whose energy is given by Eq. (4.27) with $\{n_j\} = \{\bar{n}_j\} = \{0, 0, \dots\}$. Thus, $|\Delta N, \Delta D\rangle_W$ is the state that describes a low-lying excitation without particle-hole pairs.

Let us now consider the state

$$W_{-1}^0 |\Delta N, \Delta D\rangle_W . \quad (4.34)$$

Using again Eq. (3.8), we can see that this is an eigenstate of $\tilde{\mathcal{H}}_{CS}$ with energy given by Eq. (4.27) with $\{n_j\} = \{1, 0, \dots\}$ and $\{\bar{n}_j\} = \{0, 0, \dots\}$. Thus (4.34) is a state with a right particle-hole pair at level 1. Similarly, the state

$$W_{-1}^0 \overline{W}_{-1}^0 |\Delta N, \Delta D\rangle_W \quad (4.35)$$

is an eigenstate of $\tilde{\mathcal{H}}_{CS}$ with energy given by Eq. (4.27) with $\{n_j\} = \{\bar{n}_j\} = \{1, 0, \dots\}$ and represents a state with one particle-hole pair at level 1 on the right and one particle-hole pair at level 1 on the left.

These calculations can be simply generalized to higher levels, where the structure of the states is less trivial. For example, at level (2, 0) there are two eigenstates for

$\tilde{\mathcal{H}}_{CS}$:

$$\left(W_{-2}^0 + \sqrt{\xi} W_{-1}^0 W_{-1}^0\right) |\Delta N, \Delta D\rangle_W \quad , \quad (4.36)$$

$$\left(W_{-2}^0 - \frac{1}{\sqrt{\xi}} W_{-1}^0 W_{-1}^0\right) |\Delta N, \Delta D\rangle_W \quad , \quad (4.37)$$

It is easy to see that these two states have the same energy up to order $1/N$, but actually are not degenerate due to the $1/N^2$ -term of the effective hamiltonian $\tilde{\mathcal{H}}_{CS}$. The energy of (4.36) turns out to be given exactly by Eq. (4.27) with $\{n_j\} = \{2, 0, \dots\}$ and $\{\bar{n}_j\} = \{0, 0, \dots\}$, and therefore the state (4.36) can be associated to a single particle-hole excitation of level 2 on the right. The energy of (4.37) is, instead, given by Eq. (4.27) with $\{n_j\} = \{1, 1, 0, \dots, 0\}$ and $\{\bar{n}_j\} = \{0, 0, \dots, 0\}$ and coincides with the Bethe Ansatz energy for two particle-hole excitations at level 1 on the right.

Increasing the level, one gets more states (in general at level (n, \bar{n}) we have $g(n) \times g(\bar{n})$ states, where $g(n)$ is the number of partitions of n). For instance, at level $(3, 0)$ there are three eigenstates of $\tilde{\mathcal{H}}_{CS}$, which are

$$\left(W_{-3}^0 + \frac{3\sqrt{\xi}}{2} W_{-2}^0 W_{-1}^0 + \frac{\xi}{2} W_{-1}^0 W_{-1}^0 W_{-1}^0\right) |\Delta N, \Delta D\rangle_W \quad , \quad (4.38)$$

$$\left(W_{-3}^0 + \left(\sqrt{\xi} - \frac{1}{\sqrt{\xi}}\right) W_{-2}^0 W_{-1}^0 - W_{-1}^0 W_{-1}^0 W_{-1}^0\right) |\Delta N, \Delta D\rangle_W \quad , \quad (4.39)$$

$$\left(W_{-3}^0 - \frac{3}{2\sqrt{\xi}} W_{-2}^0 W_{-1}^0 + \frac{1}{2\xi} W_{-1}^0 W_{-1}^0 W_{-1}^0\right) |\Delta N, \Delta D\rangle_W \quad , \quad (4.40)$$

and their energies are given by Eq. (4.27) with

$$\begin{aligned} \{n_j\} &= \{3, 0, 0, 0, \dots\} \quad , \quad \{\bar{n}_j\} = \{0, 0, \dots\} \quad , \\ \{n_j\} &= \{2, 1, 0, 0, \dots\} \quad , \quad \{\bar{n}_j\} = \{0, 0, \dots\} \quad , \\ \{n_j\} &= \{1, 1, 1, 0, \dots\} \quad , \quad \{\bar{n}_j\} = \{0, 0, \dots\} \quad , \end{aligned}$$

respectively.

These examples clearly support our conjecture, and show that the eigenstates of $\tilde{\mathcal{H}}_{CS}$ are in one to one correspondence with the states of the conformal Verma module at a given level. When one considers the subleading $1/N^2$ terms in addition to the leading (conformal) $1/N$ -term, formulating the model in the context of the $W_{1+\infty}$ *extended* conformal field theories, the degeneracy of the Verma module at a given level is completely removed for generic values of ξ ⁷. This piece of information is clearly important to compute the exact partition and correlation functions in the low-energy regime.

⁷ The degeneracy of the energy under the exchange $Q \leftrightarrow \bar{Q}$ and $\{n_j\} \leftrightarrow \{\bar{n}_j\}$ obviously remains.

5 The $W_{1+\infty}$ structure of the Heisenberg model in a magnetic field

The main issue of this section will be to rewrite the hamiltonian of the Heisenberg model in terms of $W_{1+\infty}$ generators as we have done in the Calogero-Sutherland case. However, it is first necessary to establish the relationship between the magnetization σ and the external field B . This can be easily obtained by requiring that the energy of the excitations vanishes on the Fermi surface, *i.e.* by equating to zero the coefficient of the operator $\sum_{r=-\infty}^{\infty} (: a_r^\dagger a_r : + : b_r^\dagger b_r :)$ in the hamiltonian (2.37). To the leading order in the $1/N$ expansion, this requirement turns into the condition

$$-B + J_z \sigma + \sin \frac{\pi \sigma}{2} + \frac{J_z}{\pi} \sin \pi \sigma = 0 \quad . \quad (5.1)$$

This equation defines the magnetization in terms of the magnetic field, and therefore fixes the position of the Fermi points $\pm n_F$ according to Eq. (2.16). It also implies that our approach is meaningful only for $|B| < B_c$, where the critical field

$$B_c = 1 + J_z$$

corresponds to magnetization $\sigma = 1$. Eq. (5.1) can be explicitly solved when $B \rightarrow 0$ and $B \rightarrow B_c$. In the first case one has

$$\sigma = \frac{2}{\pi} \left(1 + \frac{4J_z}{\pi} \right)^{-1} B \quad , \quad (5.2)$$

while in the second case one obtains

$$\sigma = 1 - \frac{2}{\pi} \sqrt{2(B_c - B)} \quad . \quad (5.3)$$

It is easy to check that these values agree, to first order in J_z , with the exact magnetization derived from the Bethe Ansatz solution of the Heisenberg model (see for instance [4]). For this agreement to occur, the last term in Eq. (5.1) is crucial. This term is produced by the two-body part of the backscattering hamiltonian, Eq. (2.36), and thus is a normal ordering effect.

We are now in the position of writing the effective hamiltonian of the Heisenberg model (2.37) in terms of the $W_{1+\infty}$ generators. Like in the Calogero-Sutherland case, here also we introduce two sets of $W_{1+\infty}$ currents, V_ℓ^i and \bar{V}_ℓ^i , represented respectively as bilinear fermionic forms in the a and b oscillators according to Eq. (3.14). Then, using Eq. (5.1), the hamiltonian (2.37) becomes

$$\mathcal{H} = \sum_{k=1}^{\infty} \left(\frac{2\pi}{N} \right)^k \mathcal{H}_{(k)} \quad , \quad (5.4)$$

where the first terms are

$$\begin{aligned}\mathcal{H}_{(1)} = & \left(\cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \cos \pi\sigma \right) (V_0^1 + \bar{V}_0^1) + \frac{J_z}{2\pi} \sum_{\ell=-\infty}^{\infty} (V_{-\ell}^0 V_{\ell}^0 + \bar{V}_{-\ell}^0 \bar{V}_{\ell}^0) \\ & + \frac{J_z}{\pi} (\cos \pi\sigma + 1) \sum_{\ell=-\infty}^{\infty} V_{\ell}^0 \bar{V}_{\ell}^0 \quad ,\end{aligned}\tag{5.5}$$

$$\begin{aligned}\mathcal{H}_{(2)} = & -\frac{1}{2} \left(\sin \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \sin \pi\sigma \right) \left[(V_0^2 + \bar{V}_0^2) - \frac{1}{12} (V_0^0 + \bar{V}_0^0) \right] \\ & - \frac{J_z}{\pi} \sin \pi\sigma \sum_{\ell=-\infty}^{\infty} (V_{\ell}^1 \bar{V}_{\ell}^0 + V_{\ell}^0 \bar{V}_{\ell}^1)\end{aligned}\tag{5.6}$$

$$\begin{aligned}\mathcal{H}_{(3)} = & -\frac{1}{6} \left(\cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \cos \pi\sigma \right) (V_0^3 + \bar{V}_0^3) \\ & + \frac{1}{10} \left(\frac{7}{12} \cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \cos \pi\sigma \right) (V_0^1 + \bar{V}_0^1) \\ & - \frac{J_z}{4\pi} \sum_{\ell=-\infty}^{\infty} \left[\ell^2 (V_{-\ell}^0 V_{\ell}^0 + \bar{V}_{-\ell}^0 \bar{V}_{\ell}^0) + \left(2\ell^2 + \frac{1}{3}(\ell^2 - 1) \cos \pi\sigma \right) V_{\ell}^0 \bar{V}_{\ell}^0 \right] \\ & - \frac{J_z}{2\pi} \cos \pi\sigma \sum_{\ell=-\infty}^{\infty} (V_{\ell}^2 \bar{V}_{\ell}^0 + 2V_{\ell}^1 \bar{V}_{\ell}^1 + V_{\ell}^0 \bar{V}_{\ell}^2) \quad .\end{aligned}\tag{5.7}$$

We remark that $\mathcal{H}_{(0)}$ vanishes due to Eq. (5.1); furthermore after inclusion of the $O(1/N^2)$ -corrections to Eq. (5.1), also the second line of Eq. (2.40) vanishes.

These equations display the effective hamiltonian of the Heisenberg model as a combination of $W_{1+\infty}$ currents. However, it is important to recall that in the absence of an external magnetic field, Umklapp terms should be also taken into account; these terms (see Eq. (2.14)) would give contributions to \mathcal{H} of the form $a_{\frac{1}{2}}^{\dagger} b_{\frac{1}{2}} a_{\frac{1}{2}}^{\dagger} b_{\frac{1}{2}}$ or $b_{\frac{1}{2}}^{\dagger} a_{\frac{1}{2}} b_{\frac{1}{2}}^{\dagger} a_{\frac{1}{2}}$, which destroy conformal invariance and cannot be written in terms of $W_{1+\infty}$ generators. This fact should not come as a surprise because it is well-known that Umklapp terms spoil charge-current conservation, which is expressed by the $W_{1+\infty}$ algebra.

We now focus on the $1/N$ -term of the effective hamiltonian given by Eq. (5.5), and proceed exactly as in the Calogero-Sutherland case. We first replace $(V_0^1 + \bar{V}_0^1)$ with a quadratic form in V_{ℓ}^0 and \bar{V}_{ℓ}^0 by means of the Sugawara construction, and then introduce new operators W_{ℓ}^0 and \bar{W}_{ℓ}^0 according to Eq. (4.6). The resulting quadratic form becomes diagonal in the new generators if, to first order in J_z , we choose ⁸

$$\tanh 2\beta = \frac{2J_z}{\pi} \cos \frac{\pi\sigma}{2} \quad .\tag{5.8}$$

⁸Of course the diagonalization of $\mathcal{H}_{(1)}$ can be done exactly to all orders in J_z ; however, as it will be clear in the following, only the first order in J_z is meaningful.

This choice implies also that W_ℓ^0 and \overline{W}_ℓ^0 are the modes of a bosonic field compactified on a circle of radius

$$r = \exp(-\beta) \simeq 1 - \frac{J_z}{\pi} \cos \frac{\pi\sigma}{2} \quad . \quad (5.9)$$

If we use once more the Sugawara construction, we finally obtain

$$\mathcal{H}_{(1/N)} \equiv \frac{2\pi}{N} \mathcal{H}_{(1)} = \frac{2\pi}{N} \left[\cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} (\cos \pi\sigma + 1) \right] (W_0^1 + \overline{W}_0^1) \quad . \quad (5.10)$$

The r.h.s. clearly exhibits the well-known fact that the effective hamiltonian of the Heisenberg model to order $1/N$ is that of a $c = 1$ conformal field theory. From Eq. (5.10), we also read that the Fermi velocity of the low-energy excitations is

$$v = \cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} (\cos \pi\sigma + 1) \quad (5.11)$$

Obviously both the Fermi velocity v and the compactification radius r depend on B through σ , and whenever Eq. (5.1) can be solved, one can obtain their explicit relation with the magnetic field. In particular, when $B \rightarrow 0$ from Eq. (5.2) we have

$$\begin{aligned} r &= \left(1 - \frac{J_z}{\pi}\right) \left(1 + \frac{J_z}{2\pi} B^2\right) \quad , \\ v &= 1 + \frac{2J_z}{\pi} \quad , \end{aligned} \quad (5.12)$$

while, when $B \rightarrow B_c$ from Eq. (5.3) we have

$$\begin{aligned} r &= 1 - \frac{J_z}{\pi} \sqrt{2(B_c - B)} \quad , \\ v &= \sqrt{2(B_c - B)} \quad . \end{aligned} \quad (5.13)$$

We observe that both v and r have the right asymptotic behavior when B is close either to the critical field or to zero, coinciding to first order in J_z with the result obtained by Bethe Ansatz (see Ref. [4]).

It is interesting to realize that the changes of r and v induced by the Bogoliubov transformation from $\{V_\ell^0, \overline{V}_\ell^0\}$ to $\{W_\ell^0, \overline{W}_\ell^0\}$ have two different origins. In fact, the compactification radius changes from $r_0 = 1$ to the value r given in Eq. (5.9), as a consequence of the left-right mixing term proportional to $V_\ell^0 \overline{V}_\ell^0$ in Eq. (5.5). This term is a marginal operator which originates both from the backward and the forward scattering hamiltonians to order $1/N$ (see Eqs. (2.35) and (2.30)). This is to be contrasted with the situation of the Calogero-Sutherland model, where only the backscattering part of the hamiltonian contributes at order $1/N$, since the forward scattering terms are $O(1/N^2)$ (see Eq. (2.54)).

The change in the Fermi velocity from $v_0 = \cos(\pi\sigma/2)$ to v given in Eq. (5.11) is, instead, due to the left and right diagonal terms proportional to J_z in $\mathcal{H}_{(1)}$. These

terms have two distinct sources: one is the diagonal part of the forward scattering hamiltonian (2.30), and the other is the two-body part of the backward scattering hamiltonian (2.36). The latter is clearly a normal ordering effect. We would like to stress that only after taking into account both kinds of terms the Fermi velocity v agrees with the value given by the Bethe Ansatz solution to first order in J_z . Notice also that

$$v_0 r_0^2 = v r^2 \quad . \quad (5.14)$$

This relation is typical of the Luttinger systems, as we mentioned in Section 4, but it is not a property of the exact Bethe Ansatz solution of the Heisenberg model. In fact, in this case Eq. (5.14) holds only to first order in J_z (see Ref. [4]). This strongly suggests that, in order to have complete agreement between the exact solution and ours, we would need at least two different scaling functions: one for the Fermi velocity and one for the compactification radius. However, the Luttinger model approach can provide only one. Thus, from now on we will limit our considerations to the first perturbative order in J_z , where this problem does not exist. This fact was already observed in Ref. [20]. However, due to a different normal ordering prescription, only the compactification radius was found to be consistent with the Bethe Ansatz solution to first order in J_z . On the contrary, the Fermi velocity (c in the notation of Ref. [20]) turned out to be different from the exact value, even to first order in J_z . The reason for this is that the backscattering hamiltonian of Ref. [20] did not require a rearrangement of the fermionic oscillators to construct normal ordered pairs, and thus did not produce a two-body part. Furthermore, not even the forward scattering processes contributed to the effective hamiltonian (see Eq. (7) of Ref. [20]). However, as we mentioned above, it is precisely due to these two J_z -dependent terms that our result for the Fermi velocity v is consistent with the exact value to first order in J_z .

The spectrum of $\mathcal{H}_{(1/N)}$ in Eq. (5.10) follows directly from the representation theory of the $c = 1$ conformal algebra. Let $|\Delta N, \Delta D\rangle_W$ be a highest weight state such that

$$\begin{aligned} W_0^0 |\Delta N; \Delta D\rangle_W &= \left(\frac{\Delta N}{2r} + r \Delta D \right) |\Delta N; \Delta D\rangle_W \\ \bar{W}_0^0 |\Delta N; \Delta D\rangle_W &= \left(\frac{\Delta N}{2r} - r \Delta D \right) |\Delta N; \Delta D\rangle_W \quad , \end{aligned} \quad (5.15)$$

with $\Delta N - 2\Delta D = 0 \pmod{2}$. Here r is the radius given by Eq. (5.9) and the numbers ΔN and ΔD have the same interpretation as in the Calogero-Sutherland case ⁹. Then, it is immediate to verify that

$$\mathcal{H}_{(1/N)} |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W = \mathcal{E}_{(1/N)} |\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W \quad (5.16)$$

⁹Note however that ΔN , being the increment in the number of fermions (*i.e.* the number of the spin-up particles) has nothing to do with N which, here, is the number of sites.

where

$$\mathcal{E}_{(1/N)} = \frac{2\pi}{N} v \left[\left(\frac{(\Delta N)^2}{4r^2} + r^2 (\Delta D)^2 + k + \bar{k} \right) \right] , \quad (5.17)$$

with $k = \sum_i k_i$ and $\bar{k} = \sum_j \bar{k}_j$ being the right and left levels of the descendant state $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W$. The energy eigenvalues (5.16) coincide with those obtained by calculating the finite size corrections from the Bethe Ansatz to first order in J_z , and are clearly degenerate when $k \geq 2$ or $\bar{k} \geq 2$.

This degeneracy is removed if we also take into account the higher order terms in the $1/N$ expansion of the effective hamiltonian (5.4). When the magnetic field is not zero, the first subleading correction is given by $\mathcal{H}_{(2)}$ of Eq. (5.6). Using the generalized Sugawara construction, and then introducing the new $W_{1+\infty}$ generators through the Bogoliubov transformation, we obtain after some straightforward algebra

$$\mathcal{H}_{(2)} = \mathcal{H}'_{(2)} + \mathcal{H}''_{(2)}$$

where

$$\begin{aligned} \mathcal{H}'_{(2)} = & -\frac{1}{2} \left(\sin \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \sin \pi\sigma \right) \left[W_0^2 + \bar{W}_0^2 - \frac{1}{12} (W_0^0 + \bar{W}_0^0) \right] \\ & - \frac{J_z}{2\pi} \sin \pi\sigma \left(W_0^0 \bar{W}_0^1 + W_0^1 \bar{W}_0^0 \right) , \end{aligned} \quad (5.18)$$

and

$$\mathcal{H}''_{(2)} = -\frac{J_z}{2\pi} \sin \pi\sigma \sum_{\ell \neq 0} \left(W_\ell^0 \bar{W}_\ell^1 + W_\ell^1 \bar{W}_\ell^0 \right) . \quad (5.19)$$

Neither $\mathcal{H}'_{(2)}$ nor $\mathcal{H}''_{(2)}$ are diagonal in the basis $|\Delta N, \Delta D; \{k_i\}, \{\bar{k}_j\}\rangle_W$ previously considered. However, like in the Calogero-Sutherland model, we can find suitable combinations of these states that diagonalize simultaneously $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$. In fact, these two operators commute with each other, since they are combinations of the generators of the Cartan subalgebra of $W_{1+\infty}$. Notice that $\mathcal{H}'_{(2)}$ is not factorized, but contains a left-right mixing term. This is a new feature that distinguishes between the Heisenberg and the Calogero-Sutherland models. However, as we shall see in a moment, this mixing does not cause any problem. Let us denote by $\mathcal{W}(\Delta N, \Delta D)$ the set of states that are simultaneously eigenstates of $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$. It is easy to check that $\mathcal{H}''_{(2)}$ given in Eq. (5.19) has vanishing expectation value on any state of $\mathcal{W}(\Delta N, \Delta D)$. Therefore, according to ordinary perturbation theory, $\mathcal{H}''_{(2)}$ gives no contribution to first order in J_z , and can be dropped. The effective hamiltonian of the Heisenberg model up to $O(1/N^2)$ is, then,

$$\mathcal{H}_H \equiv \mathcal{H}_{(1/N)} + \left(\frac{2\pi}{N} \right)^2 \mathcal{H}'_{(2)}$$

$$\begin{aligned}
&= \frac{2\pi}{N} \left[\cos \frac{\pi\sigma}{2} + \frac{J_z}{\pi} (\cos \pi\sigma + 1) \right] (W_0^1 + \overline{W}_0^1) \\
&\quad - \left(\frac{2\pi}{N} \right)^2 \left\{ \frac{1}{2} \left(\sin \frac{\pi\sigma}{2} + \frac{J_z}{\pi} \sin \pi\sigma \right) \left[W_0^2 + \overline{W}_0^2 - \frac{1}{12} (W_0^0 + \overline{W}_0^0) \right] \right. \\
&\quad \left. + \frac{J_z}{2\pi} \sin \pi\sigma (W_0^0 \overline{W}_0^1 + W_0^1 \overline{W}_0^0) \right\} . \tag{5.20}
\end{aligned}$$

and its spectrum can be easily found. Actually, one can verify that the eigenstates of \mathcal{H}_H have the same form as those of the Calogero-Sutherland model with $\xi = 1$ (see for example Eqs. (4.34)-(4.40)). We denote these states simply by $|\{n_j\}, \{\overline{n}_j\}\rangle$, where $\{n_j\}$ and $\{\overline{n}_j\}$ are the numbers that specify the levels of the right and left particle-hole pairs created on the highest weight state labeled by ΔN and ΔD . Then, a direct calculation shows that

$$\begin{aligned}
W_0^0 |\{n_j\}, \{\overline{n}_j\}\rangle &= Q |\{n_j\}, \{\overline{n}_j\}\rangle , \\
W_0^1 |\{n_j\}, \{\overline{n}_j\}\rangle &= \left(\frac{1}{2} Q^2 + \sum_{j=1} n_j \right) |\{n_j\}, \{\overline{n}_j\}\rangle , \\
W_0^2 |\{n_j\}, \{\overline{n}_j\}\rangle &= \left(\frac{1}{3} Q^3 + 2Q \sum_{j=1} n_j + \sum_{j=1} n_j^2 - \sum_{j=1} (2j-1) n_j \right) |\{n_j\}, \{\overline{n}_j\}\rangle , \tag{5.21}
\end{aligned}$$

with Q being the right charge given by Eq. (5.15). Obviously the states $|\{n_j\}, \{\overline{n}_j\}\rangle$ are eigenstates of \overline{W}_0^0 , \overline{W}_0^1 and \overline{W}_0^2 also, and their eigenvalues are given by Eq. (5.21) with Q and $\{n_j\}$ replaced by \overline{Q} and $\{\overline{n}_j\}$, respectively. Using this result in Eq. (5.20), it is trivial to compute the energy of the low-lying excitations up to order $1/N^2$. In this case, however, the comparison with the exact results from the Bethe Ansatz is not immediate because only the leading finite size corrections to the energy are currently available in the literature.

We conclude by pointing out that in the absence of magnetic field, $\mathcal{H}_{(2)}$ is zero and the first non vanishing corrections to the conformal hamiltonian are of order $1/N^3$. If $B = 0$, these are

$$\begin{aligned}
\mathcal{H}_{(3)} &= -\frac{1}{6} \left(1 + \frac{J_z}{\pi} \right) (V_0^3 + \overline{V}_0^3) + \frac{1}{10} \left(\frac{7}{12} + \frac{J_z}{\pi} \right) (V_0^1 + \overline{V}_0^1) \\
&\quad - \frac{J_z}{4\pi} \sum_{\ell=-\infty}^{\infty} \left[\ell^2 (V_{-\ell}^0 V_{\ell}^0 + \overline{V}_{-\ell}^0 \overline{V}_{\ell}^0) + \left(2\ell^2 + \frac{1}{3}(\ell^2 - 1) \right) V_{\ell}^0 \overline{V}_{\ell}^0 \right] \\
&\quad - \frac{J_z}{2\pi} \sum_{\ell=-\infty}^{\infty} (V_{\ell}^2 \overline{V}_{\ell}^0 + 2V_{\ell}^1 \overline{V}_{\ell}^1 + V_{\ell}^0 \overline{V}_{\ell}^2) . \tag{5.22}
\end{aligned}$$

After performing the Bogoliubov transformation, $\mathcal{H}_{(3)}$ becomes

$$\mathcal{H}_{(3)} = \mathcal{H}'_{(3)} + \mathcal{H}''_{(3)} ,$$

with

$$\begin{aligned}\mathcal{H}'_{(3)} = & -\frac{1}{6}\left(1 + \frac{J_z}{\pi}\right)\left(W_0^3 + \overline{W}_0^3\right) + \frac{1}{10}\left(\frac{7}{12} + \frac{J_z}{\pi}\right)\left(W_0^1 + \overline{W}_0^1\right) \\ & - \frac{J_z}{4\pi} \sum_{\ell=-\infty}^{\infty} \ell^2 \left(W_{-\ell}^0 W_{\ell}^0 + \overline{W}_{-\ell}^0 \overline{W}_{\ell}^0\right) - \frac{J_z}{\pi} W_0^1 \overline{W}_0^1 \quad ,\end{aligned}\quad (5.23)$$

and

$$\mathcal{H}''_{(3)} = -\frac{J_z}{4\pi} \sum_{\ell \neq 0} \left[\left(2\ell^2 + \frac{1}{3}\ell^2\right) W_{\ell}^0 \overline{W}_{\ell}^0 + 4 W_{\ell}^1 \overline{W}_{\ell}^1 \right] \quad . \quad (5.24)$$

The procedure is now identical to that previously discussed. In fact, one can easily realize that $\mathcal{H}'_{(3)}$ in Eq. (5.23) and $\mathcal{H}_{(1/N)}$ in Eq. (5.10) with $\sigma = 0$ commute with each other, so that they can be diagonalized simultaneously. If we take any common eigenstate of these operators, then we can check that $\mathcal{H}''_{(3)}$ in Eq. (5.24) has always vanishing expectation value on it. Thus, according to ordinary perturbation theory, $\mathcal{H}''_{(3)}$ can be dropped to first order in J_z . Therefore, the effective hamiltonian of the Heisenberg model with no magnetic field up to order $1/N^3$ is $\mathcal{H}_{(1/N)} + (2\pi/N)^3 \mathcal{H}'_{(3)}$.

6 Conclusions

We conclude by commenting on some of the most relevant and general features of our method. First of all, we would like to stress that our algebraic bosonization can be applied to any (abelian) gapless fermionic hamiltonian consisting of a bilinear kinetic term and an arbitrary interaction. No special requirements on the form of the dispersion relation and the potential are needed. In particular, it is not necessary for the system to be integrable. In lattice models, one limitation is that no Umklapp terms should appear in the low-energy hamiltonian. Indeed, these would spoil the charge-current conservation, which is the origin of the $W_{1+\infty}$ algebraic structure of the effective theory.

Since the Fermi surface is identified from the bilinear part of the hamiltonian, our procedure is strictly perturbative in the coupling constant g of the interaction term. Limiting our analysis to the conformal leading order in the $1/N$ expansion, it is possible to diagonalize non-perturbatively the hamiltonian by means of a Bogoliubov transformation. However, once we also include the subleading $O(1/N^2)$ -part of the effective hamiltonian, only the first perturbative order in the coupling constant is meaningful. In fact, when one also takes into account the $O(1/N^2)$ -terms of the dispersion curve around the Fermi surface, spurious states are effectively introduced. These would contribute beyond the first perturbative order, spoiling the finiteness of

the theory. In some cases, however, non-perturbative improvements are possible. For example, by exploiting some results of the Bethe Ansatz solution of the Calogero-Sutherland model, we have been able to write the *complete* effective hamiltonian for *any* value of the coupling constant (see Eq. (4.33)), and compute its low-energy spectrum using purely algebraic methods. However, even if such improvements are possible, for theories with a non-trivial phase diagram we can only hope to reach the phase continuously connected to $g = 0$.

Finally, we point out that the complete effective hamiltonian does not show, in general, a factorization between the left and right sectors, contrarily what happens at the leading conformal order (see Eq. (1.1)). However, the left-right mixing can only occur through the zero modes of the generators $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra (see for example the hamiltonian of the Heisenberg model given in Eq. (5.20)). Hence, it is still possible to use the representation theory of the chiral $W_{1+\infty}$ algebra to compute the low-energy spectrum of the model.

Acknowledgements

We would like to thank the organizers of the Benasque Center for Physics for their kind hospitality during the early stages of this work.

G.R.Z. thanks the Dipartimento di Fisica Teorica dell' Università di Torino and I.N.F.N. (sezione di Torino) for support and hospitality during the period in which most part of this work has been done.

This research was partially supported by MURST and the EU, within the framework of the program “Gauge Theories, Applied Supersymmetry and Quantum Gravity”, under contract SCI*-CT92-0789.

A The mathematics of the $W_{1+\infty}$ algebra

In this appendix we collect several mathematical results concerning the $W_{1+\infty}$ algebra, in particular we give the complete expression of the algebra in a compact form, a survey of representation theory and briefly discuss the characters of the representations. All of these results are taken from the original references [13, 21, 28]. We also quote here some formulae regarding the expression of the $W_{1+\infty}$ generators on the conformal plane and on the cylinder.

The complete form of the $W_{1+\infty}$ algebra

The complete $W_{1+\infty}$ algebra is expressed in compact form by using a parametric sum of the V_n^i current modes, denoted by $V(-z^n \exp(\lambda D))$, where $D \equiv z \frac{\partial}{\partial z}$ [13]. These satisfy the algebra,

$$\begin{aligned} [V(-z^r e^{\lambda D}), V(-z^s e^{\mu D})] &= (e^{\mu r} - e^{\lambda s}) V(-z^{r+s} e^{(\lambda+\mu)D}) \\ &\quad + c \delta_{r+s,0} \frac{e^{-\lambda r} - e^{-\mu s}}{1 - e^{\lambda+\mu}}, \end{aligned} \quad (\text{A.1})$$

where c is the central extension. The currents V_n^i of conformal spin $h = i + 1 \geq 1$ and mode index $n \in \mathbf{Z}$, are identified by expanding this parametric operator in λ , namely

$$V_n^i \equiv V(-z^n f_n^i(D)), \quad (\text{A.2})$$

where $f_n^i(D)$ are specific i -th order polynomials which diagonalize the central term of Eq. (A.1) in the i, j indices [13, 17]. For example, we have

$$\begin{aligned} V_n^0 &\equiv V(-z^n), \\ V_n^1 &\equiv V\left(-z^n \left(D + \frac{n+1}{2}\right)\right), \\ V_n^2 &\equiv V\left(-z^n \left(D^2 + (n+1)D + \frac{(n+1)(n+2)}{6}\right)\right), \\ V_n^3 &\equiv V\left(-z^n \left(D^3 + \frac{3}{2}(n+1)D^2 + \frac{(6n^2 + 15n + 11)}{10}D \right. \right. \\ &\quad \left. \left. + \frac{(n+1)(n+2)(n+3)}{20}\right)\right). \end{aligned} \quad (\text{A.3})$$

A survey on representation theory

All unitary irreducible quasi-finite highest-weight representations [21], denoted by $M(W_{1+\infty}, c, \vec{Q})$, exist when the central charge $c = m$ is a positive integer, and are characterized by a highest weight state $|\vec{Q}\rangle_W$, which satisfies

$$V(-z^n e^{\lambda D}) |\vec{Q}\rangle_W = 0 \quad , \quad n > 0 \quad , \quad (\text{A.4})$$

and

$$V(-e^{\lambda D})|\vec{Q}\rangle_W = \Delta(\lambda)|\vec{Q}\rangle_W \equiv \sum_{i=1}^m \frac{e^{\lambda Q_i} - 1}{e^\lambda - 1} |\vec{Q}\rangle_W \quad , \quad (\text{A.5})$$

where $\vec{Q} = \{Q_1, \dots, Q_m\} \in \mathbf{R}^m$. In particular, the eigenvalues of the operators V_0^0 and V_0^1 given in Eq. (3.8) can be recovered by expanding $\Delta(\lambda)$ and comparing to Eq. (A.3). The infinite tower of states (Verma module) in each representation is generated by expanding in the $\{\lambda_i\}$ of

$$V(-z^{-n_1} e^{\lambda_1 D}) \dots V(-z^{-n_k} e^{\lambda_k D}) |\vec{Q}\rangle_W \quad , \quad n_1 \geq n_2 \dots \geq n_k > 0 \quad , \quad (\text{A.6})$$

where $n = \sum_{i=1}^k n_i$ is the level of the states. The quasi-finite representations have only a finite number of independent states at each level, thus there are an infinity of polynomial relations among the generators V_n^i , whose explicit form depends on the values of c and \vec{Q} . The number of independent states $d(n)$ at level n is encoded in the (specialized) character of the representation ($|q| < 1$) [8],

$$\chi_{M(W_{1+\infty}, m, \vec{Q})}(q) \equiv \text{tr}_{M(W_{1+\infty}, m, \vec{Q})} (q^{V_0^1 - \frac{m}{24}}) = q^{\sum_{i=1}^m (\frac{Q_i^2}{2} - \frac{1}{24})} \sum_{n=0}^{\infty} d(n) q^n \quad . \quad (\text{A.7})$$

A representation is called *generic* if the weight \vec{Q} has components $(Q_i - Q_j) \notin \mathbf{Z}$, $\forall i \neq j$, and *degenerate* if it has $(Q_i - Q_j) \in \mathbf{Z}$ for some $i \neq j$. The weight components $\{Q_i\}$ of the degenerate representations can be grouped and ordered in congruence classes modulo \mathbf{Z} [21],

$$\begin{aligned} \{Q_i, \dots, Q_m\} &= \{s_1 + n_1^{(1)}, \dots, s_1 + n_{m_1}^{(1)}\} \cup \dots \cup \{s_k + n_1^{(k)}, \dots, s_k + n_{m_k}^{(k)}\} \quad , \\ n_j^{(i)} &\in \mathbf{Z} \quad , \quad n_1^{(i)} \geq n_2^{(i)} \geq \dots \geq n_{m_i}^{(i)} \quad , \quad m = \sum_{i=1}^k m_i \quad , \quad s_i \in \mathbf{R} \quad . \end{aligned} \quad (\text{A.8})$$

A two-class representation is the tensor product of two one-class representations. Therefore, the one-class degenerate representations are the basic building blocks, which one can use to construct the $W_{1+\infty}$ minimal models [28]. The character for the generic representations is

$$\chi_{M(W_{1+\infty}, m, \vec{Q})}(q) = \prod_{i=1}^m \frac{q^{Q_i^2/2}}{\eta(q)} = \prod_{i=1}^m \chi_{M(\widehat{U(1)}, 1, Q_i)}(q) \quad , \quad (\text{A.9})$$

where $\eta(q)$ is the Dedekind function,

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad . \quad (\text{A.10})$$

Eq. (A.9) exhibits the form of the $W_{1+\infty}$ character in terms of m characters of the $\widehat{U(1)}$ algebra, which implies a one-to-one equivalence of generic $W_{1+\infty}$ and $\widehat{U(1)}^{\otimes m}$

representations [21, 28]. The character for the one-class degenerate representations is

$$\begin{aligned} \chi_{M(W_{1+\infty}, m, \vec{Q})}(q) &= \eta(q)^{-m} q^{\sum_{i=1}^m Q_i^2/2} \prod_{1 \leq i < j \leq m} (1 - q^{n_i - n_j + j - i}) \quad , \\ \vec{Q} = \{Q_1, \dots, Q_m\} &= \{s + n_1, \dots, s + n_m\} \quad , \quad n_1 \geq \dots \geq n_m \quad . \end{aligned} \quad (\text{A.11})$$

Note that the number of independent states $d(n)$ at level n is smaller for degenerate (A.11) than for generic (A.7) representations, because the former have additional relations among the states that lead to null vectors. This is the origin of reducibility of the $\widehat{U(1)}^{\otimes m}$ representations with respect to the $W_{1+\infty}$ algebra.

In order to construct a $W_{1+\infty}$ theory, one should combine $W_{1+\infty}$ representations that are closed under the fusion rules [8]. For *generic* $W_{1+\infty}$ representations, the fusion rules require that all highest weight vectors \vec{Q} span a lattice Γ ,

$$\Gamma = \left\{ \vec{Q} \mid \vec{Q} = \sum_{i=1}^m n_i \vec{v}_i \quad , \quad n_i \in \mathbf{Z} \right\} \quad , \quad (\text{A.12})$$

whose points satisfy $(Q_i - Q_j) \notin \mathbf{Z}$, $\forall i \neq j$ (see [28] for more details). The resulting $W_{1+\infty}$ theory can be associated to a system with m components at the Fermi point with the basis vector \vec{v}_i representing a physical elementary excitation in the i -th component of the Fermi point.

The $W_{1+\infty}$ operators on the cylinder

It is well-known that the form (3.15) of the $W_{1+\infty}$ currents on the plane, constructed out of the operators (3.12), is different from that of the physical currents on the cylinder, constructed out of the operators (2.27) and (2.28). This is due to a normal ordering effect (for a review see Ref. [29]). To obtain their specific form on the cylinder in the fermionic case ($c = 1$), one applies the conformal mapping (3.11) to each fermion field in (3.15), paying attention to the different normal ordering in the plane and the cylinder (for a detailed account see [17]). For example,

$$\begin{aligned} V_R^0(u) &= : F^\dagger(u) F(u) : \equiv \lim_{u_1, u_2 \rightarrow u} \left(F^\dagger(u_1) F(u_2) - \frac{1}{u_1 - u_2} \right) \\ &= \frac{dz}{du} V^0(z) + \lim_{u_1, u_2 \rightarrow u} \left[\left(\frac{dz_1}{du_1} \frac{dz_2}{du_2} \right)^{1/2} \frac{1}{z_1 - z_2} - \frac{1}{u_1 - u_2} \right] \\ &= \frac{z}{R} V^0(z) \quad . \end{aligned} \quad (\text{A.13})$$

Proceeding similarly for the other currents, one can obtain the explicit relations between the zero modes in the two geometries; for the first few values of the spin these are

$$(V_R)_0^0 = V_0^0 \quad ,$$

$$\begin{aligned}
(V_R)_0^1 &= \frac{1}{R} \left(V_0^1 - \frac{1}{24} \right) , \\
(V_R)_0^2 &= \frac{1}{R^2} \left(V_0^2 - \frac{1}{12} V_0^0 \right) , \\
(V_R)_0^3 &= \frac{1}{R^3} \left(V_0^3 - \frac{7}{20} V_0^1 - \frac{7}{960} \right) .
\end{aligned} \tag{A.14}$$

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